Available online at www.sciencedirect.com

SciVerse ScienceDirect

Stochastic Processes and their Applications 122 (2012) 1456–1486

stochastic processes and their applications

www.elsevier.com/locate/spa

Scalar conservation laws with fractional stochastic forcing: Existence, uniqueness and invariant measure☆

Bruno Saussereau[a,*], Ion Lucretiu Stoica[b]

[a] *Laboratoire de Mathématiques de Besançon, CNRS, UMR 6623, 16 Route de Gray, 25030 Besançon cedex, France*
[b] *Institute of Mathematics, "Simion Stoilow" of the Romanian Academy and Faculty of Mathematics, University of Bucharest, Str. Academiei 14, Bucharest RO-70109, Romania*

Received 18 February 2011; received in revised form 21 October 2011; accepted 4 January 2012
Available online 11 January 2012

Abstract

We study a fractional stochastic perturbation of a first-order hyperbolic equation of nonlinear type. The existence and uniqueness of the solution are investigated via a Lax–Oleĭnik formula. To construct the invariant measure we use two main ingredients. The first one is the notion of a generalized characteristic in the sense of Dafermos. The second one is the fact that the oscillations of the fractional Brownian motion are arbitrarily small for an infinite number of intervals of arbitrary length.

Keywords: Scalar conservation laws; Random perturbations; Variational principle; Deterministic control theory; Hamilton–Jacobi–Bellman equation; Fractional Brownian motion

1. Introduction

In this paper we study the following scalar conservation law:

$$\partial_t u(t,x,\omega) + \partial_x \Psi\big(u(t,x,\omega)\big) = \partial_x \dot{F}(t,x,\omega). \tag{1}$$

In the above equation, $x \in \mathbb{R}$, $t \geq t_0$, $u(t,x,\cdot)$ is a random variable with values in $\mathbb{R}$ and F is a random force. A deterministic initial datum $u(t_0,x) = u_0(x)$ is given at a fixed time t_0 and we assume that $u_0 \in L^\infty(\mathbb{R})$. As usual, the random force will not be differentiable in the time

☆ This work was partially support by the Laboratoire Européen Associé CNRS Franco–Roumain, Math Mode.

* Corresponding author.
E-mail addresses: bruno.saussereau@univ-fcomte.fr (B. Saussereau), lstoica@fmi.unibuc.ro (I.L. Stoica).

0304-4149/$ - see front matter
doi:10.1016/j.spa.2012.01.005

variable; hence $\dot{F}$ denotes its formal time derivative. The sense given to the above equation will be stated below using a weak formulation.

When the random force F is null, Eq. (1) is a deterministic scalar conservation law and there is a wide literature on this subject. We recall that the weak solution to such a deterministic problem is not unique in general. One needs to introduce the notion of an entropy solution in order to identify the physical solution. Furthermore the selected solution has a nice qualitative behavior: discontinuities that are related to the creation of shocks, and a description of the behavior in terms of characteristics (see [1]). The books in the non-exhaustive list [2,5,10,16] provide didactic introductions to this subject.

Stochastic scalar conservation laws constitute a topic that has been of growing interest in the past few years. Nevertheless, there are only a few works on this subject. In [9] an operator splitting method is proposed for proving the existence of a weak solution to the Cauchy problem $du + \partial_x f(u)dt = g(u)dW_t$ for $x \in \mathbb{R}$. In [12] a method of compensated compactness is used to prove the existence of a stochastic weak entropy solution to the problem $du + \partial_x f(u)dt = g(t,x)dW_t,\ x \in \mathbb{R}$. The uniqueness is achieved using a Kruzkhov-type method. A notion of a strong entropy solution is proposed by [6] in order to extend the above-mentioned result to the problem $du + \operatorname{div} f(u)dt = \sigma(t,u)dW_t,\ x \in \mathbb{R}^d$. A stochastic scalar conservation law in a bounded domain of $\mathbb{R}^d$ is investigated in [19] using a measure-valued solution and Kruzkhov's entropy formulation. Finally in [3] it is proved that the Cauchy problem for a randomly forced, periodic multi-dimensional scalar first-order conservation law with additive or multiplicative noise admits a unique solution, characterized by a kinetic formulation of the problem.

Besides these works, the paper of E et al. [20] is the starting point of our investigation. This article deals with Burgers' case (that is $\Psi(u) = u^2/2$):

$$\partial_t u(t,x,\omega) + \partial_x \big(u(t,x,\omega)\big)^2 = \partial_x \dot{F}(t,x,\omega),$$

with a stochastic forcing given by $F(t,x,\omega) = \sum_{k=1}^{\infty} F_k(x)\dot{B}_k(t)$ where $(B_k)_{k\geq 1}$ are independent standard Wiener processes on the real line $\mathbb{R}$ ($\dot{B}_k$ is again the formal time derivative of this process). The existence and uniqueness are proved, together with the existence of an invariant measure. A parabolic perturbation problem approach is considered, based on the Hopf–Cole transformation.

On the one hand, our work is a generalization of the existence and uniqueness results contained in [20]: we work with a general conservation law depending on the function Ψ and we can also reach a large noise class having Hölder continuous trajectories. A Lax–Oleĭnik formula is given using a direct approach via the Hamilton–Jacobi equation that is naturally associated with our problem. The existence and uniqueness result is presented in the next section (see Theorem 1).

On the other hand, we generalize the existence of an invariant measure to the case of a fractional noise when the sequence of independent Brownian motions is replaced by fractional Brownian motions (fBm for short) on the real line. There are serious difficulties in working with fBm. First, unlike the classical Brownian motion, the two-sided fBm is not obtained by gluing two independent copies of a one-sided fBm together at time $t = 0$. Moreover, when $t \leq 0$, the two-sided fBm is no longer a Volterra-type process (see [11] for a more detailed discussion of this fact). In [20], there is roughly speaking only one purely probabilistic property of the noise that is employed: the Brownian noise is arbitrarily small on an infinite number of arbitrary long time intervals. In other words for all $\varepsilon > 0, T > 0$, for almost all ω, there exists a sequence of random times $(t_n(\omega))_{n\geq 1}$ such that $t_n(\omega) \to -\infty$ and

$$\forall n, \qquad \sup_{t_n - T \leq s \leq t_n} \sum_{k\geq 1} \Big\{ \|F_k\|_{C_b^2(\mathbb{R})} |B_k(s) - B_k(t_n)| \Big\} \leq \varepsilon.$$

This result relies on the independence of the increments of a Brownian motion and on the Borel–Cantelli lemma. In a fractional Brownian framework the increments are no longer independent. We will be able to adapt this argument thanks to a (reversed) conditional version of the Borel–Cantelli lemma. The analogous property for the trajectories of a fBm is new as far as we are aware.

In the following section, we state our hypothesis and we give the main results of our work. Section 3 is devoted to the variational principle which is used to prove the existence and uniqueness. As regards the calculus of variations problem considered in Section 3, we study in Section 4 a particular class of minimizers of the action appearing in the Lax–Oleĭnik formula: the one-sided minimizers. They are used to construct a unique solution of (1) defined on the time interval $\mathbb{R}$ in such a way that the random attractor consists of a single trajectory. Then we prove easily the existence of an invariant solution. Finally, the proof of the oscillation property (see Theorem 2) of the fractional noise is given in Section 5. Some technical proofs appear in the Appendix.

2. Notation and the main results

We will use the following notation:

- $C_b^r(\mathbb{R})$ is the space of bounded functions that are differentiable r times with bounded derivatives endowed with the norm given by $\|\varphi\|_{C_b^r(\mathbb{R})} = \sum_{i=0}^r \|\varphi^{(i)}\|_\infty$;
- for $0 < \lambda < 1$ and $-\infty < a < b < +\infty$, $C^\lambda(a, b)$ is the space of λ-Hölder continuous functions $f : [a, b] \to \mathbb{R}$, equipped with the norm $\|f\|_\lambda := \|f\|_{a,b,\infty} + \|f\|_{a,b,\lambda}$, where

$$\|f\|_{a,b,\infty} = \sup_{a\leq r\leq b} |f(r)| \quad \text{and} \quad \|f\|_{a,b,\lambda} = \sup_{a\leq r\leq s\leq b} \frac{|f(s) - f(r)|}{|s - r|^\lambda};$$

- for two times t_1, t_2, $H^1(t_1, t_2)$ is the Sobolev space of $L^2(t_1, t_2)$-weakly differentiable functions from $[t_1, t_2]$ to $\mathbb{R}$ equipped with the scalar product

$$\langle \xi_1, \xi_2 \rangle = \int_{t_1}^{t_2} \xi_1(s)\xi_2(s)ds + \int_{t_1}^{t_2} \dot{\xi}_1(s)\dot{\xi}_2(s)ds;$$

- for a function f from $\mathbb{R} \to \mathbb{R}$, we denote as f^* its Legendre transform defined as $f^*(q) = \sup_{p\in\mathbb{R}} \big(pq - f(p)\big)$ for $q \in \mathbb{R}$.

In the probabilistic framework of $(\Omega, \mathcal{F}, \mathbb{P})$, we make the following assumption for the stochastic forcing term F.

Hypothesis I. For any t, x, $F(t, x) = \sum_{k=1}^\infty F_k(x)B_k(t)$ where:

(a) for any k, F_k belongs to $C_b^3(\mathbb{R})$;
(b) there exists $\lambda > 0$ such that the sequence of processes $\big((B_k(t))_{t\in(-\infty,\infty)}\big)_{k\geq 1}$ satisfies $B_k(\cdot) \in C^\lambda(a, b)$ for any $k \geq 1$, $-\infty < a < b < +\infty$;
(c) one has to impose additionally that $\sum_{k\geq 1} \|F_k\|_{C_b^3(\mathbb{R})} \|B_k\|_\lambda < \infty$.

We remark that the processes B_k are not necessarily independent. It is quite straightforward to see that the above noise term covers that of [20] but it also covers sequences of processes such as the fractional Brownian motion of any Hurst parameter.

The function Ψ will satisfy the following assumption.

Hypothesis II. The flux Ψ satisfies:

(a) Ψ is uniformly convex: there exists $\theta > 0$ such that $\Psi''(v) \geq \theta$ for all $v \in \mathbb{R}$,
(b) the super-linear growth condition: there exist $k_2 > k_1 > 0$ and two constants l_1, l_2 such that $l_1|v|^{k_1} \leq \frac{\Psi(v)}{|v|} \leq l_2|v|^{k_2}$,
(c) there exists L such that $|\Psi'(v) - \Psi'(v')| \leq L|v - v'|$,
(d) there exists a positive function $R \mapsto C(R)$ such that $|\Psi^*(v) - \Psi^*(v')| \leq C(R)|v - v'|$ whenever $\max(|v|, |v'|) \leq R$.

We stress the fact that our assumptions for Ψ are clearly true if the flux is the square function as in Burgers' case.

Now we give the precise meaning of (1).

Definition 1. A random field u defined on $[t_0, +\infty) \times \mathbb{R} \times \Omega$ with real values is a weak solution of (1) with initial condition $u(t_0, \cdot) = u_0(\cdot) \in L^\infty(\mathbb{R})$ if:

(i) For all $t > t_0$ and $x \in \mathbb{R}$, $u(t, x, \cdot)$ is measurable with respect to $\mathcal{F}_{t_0,t} = \sigma\{B_k(s), t_0 \leq s \leq t, k \geq 1\}$.
(ii) Almost surely, $u(\cdot, \cdot, \omega) \in L^1_{\text{loc}}([t_0, \infty) \times \mathbb{R})$ and $u(t, \cdot, \omega) \in L^\infty(\mathbb{R})$ for any $t \geq t_0$.
(iii) For all test functions $\varphi \in C_c^2(\mathbb{R} \times \mathbb{R})$ (the set of functions that are twice differentiable with compact support) the following equality holds almost surely:

$$\begin{aligned}
&\int_{t_0}^{\infty} \int_{\mathbb{R}} \frac{\partial \varphi(t, x)}{\partial t} u(t, x) dx dt + \int_{t_0}^{\infty} \int_{\mathbb{R}} \frac{\partial \varphi(t, x)}{\partial x} \Psi\big(u(t, x)\big) dx dt \\
&\quad = - \int_{\mathbb{R}} u_0(x) \varphi(t_0, x) dx \\
&\qquad - \int_{\mathbb{R}} \sum_{k=1}^{\infty} \left\{ F_k(x) \int_{t_0}^{\infty} \frac{\partial^2 \varphi(t, x)}{\partial t \partial x} (B_k(t) - B_k(t_0)) dt \right\} dx. \qquad (2)
\end{aligned}$$

The stochastic term appears in the above weak formulation in an unusual way. We will give some comments concerning this in Section 3.

It is well known that the notion of a weak solution is not sufficient for having uniqueness for the solution of (1) in the deterministic case. One has to introduce admissible solutions (or entropy weak solutions).

Definition 2. We say that a random field u which is already a weak solution of Eq. (1) is an entropy weak solution if there exists $C > 0$ such that for almost all $\omega \in \Omega$,

$$u(t, x + z, \omega) - u(t, x, \omega) \leq C \left(1 + \frac{1}{t - t_0} \right) z \qquad (3)$$

for all $(t, x) \in (t_0, \infty) \times \mathbb{R}$ and $z > 0$.

The above entropy condition is the historical "*condition* E", so called in [15]. It ensures the uniqueness of bounded weak solutions. It follows from (3) that for $t > t_0$ the function $x \mapsto u(t, x) - C\,x$ is nonincreasing, and consequently has left and right hand limits at each point.

Thus also $x \mapsto u(t,x)$ has left and right hand limits at each point, with $u(t,x-) \geq u(t,x+)$. So the classical form of the entropy condition holds at any point of discontinuity.

First, we are interested in the existence and uniqueness of the entropy weak solution of (1). We generalize the existence and uniqueness result of [20] for a general flux and a wide class of noise in the following theorem.

Theorem 1. *We assume Hypotheses I and II. Let $u_0 \in L^\infty(\mathbb{R})$. There exists a unique entropy weak solution to the stochastic scalar conservation law (1) such that $u(t_0,x) = u_0(x)$. For $t \geq t_0$, this solution is given by the following Lax–Oleĭnik-type formula:*

$$u(t,x,\omega) = \frac{\partial}{\partial x}\left(\inf_{\substack{\xi \in H^1(t_0,t) \\ \xi(t)=x}} \left\{ \mathcal{A}_{t_0,t} + \int_0^{\xi(t_0)} u_0(z)dz \right\}\right), \tag{4}$$

with

$$\begin{aligned}\mathcal{A}_{t_0,t}(\xi) = &\int_{t_0}^t \left\{ \Psi^*(\dot{\xi}(s)) - \sum_{k\geq 1}\big(B_k(s) - B_k(t_0)\big) F_k'(\xi(s))\dot{\xi}(s) \right\} ds \\ &+ \sum_{k\geq 1}\big(B_k(t) - B_k(t_0)\big) F_k(\xi(t)).\end{aligned} \tag{5}$$

The fact that F is random and can be decomposed as a linear combination of Hölder continuous processes plays no role in the proof of this theorem. Nevertheless, we keep this formulation for two reasons. The first one is to suit the framework of [20] in which this decomposition was essential. Indeed they use a regularization of the Brownian noise to prove the existence and uniqueness. The arguments presented here are quite different. The second reason will become clear when we deal with invariant measure (we will assume that our noise is a combination of fractional Brownian motions).

Certainly the most important contribution of our work is the study of the invariant measure for the stochastic conservation law (1) for the particular case of a fractional noise. There is only the work of E et al. available dealing with invariant measure stochastic scalar conservation laws (in the case of Burgers' equation with a Brownian noise). In order to state the results concerning the invariant measure, we work with the following particular noise term F.

Hypothesis III. For any t, x, $F(t,x) = \sum_{k=1}^\infty F_k(x)B_k(t)$ with:

(a) for any k, F_k belongs to $C_b^3(\mathbb{R})$ and $\sum_{k\geq 1} k^{2/H}\|F_k\|_{C_b^3(\mathbb{R})} < \infty$;

(b) the sequence of processes $\big((B_k(t))_{t\in\mathbb{R}}\big)_{k\geq 1}$ is a sequence of independent fractional Brownian motions with Hurst parameter $H \in (0,1)$.

We recall that $(B_k(t))_{t\in\mathbb{R}}$ being a fBm means that it is a centered Gaussian process satisfying $B_k(0) = 0$ and $\mathbb{E}(|B_k(t) - B_k(s)|^2) = |t-s|^{2H}$.

The technique that we employed to solve the problem of the existence of an invariant measure are essentially contained in [20]. Nevertheless, the probabilistic property of the noise that is employed to construct the invariant measure is the fact that it has periods of arbitrary length and arbitrary small amplitude oscillation as time goes to $-\infty$. The result, which is interesting in itself, is new in the case of a fBm:

Theorem 2. *For all $\varepsilon > 0, T > 0$, for almost all ω, there exists a sequence of random times $(t_n(\omega))_{n\geq 1}$ such that $t_n(\omega) \to -\infty$ and*

$$\forall n, \quad \sum_{k\geq 1}\Big\{\|F_k\|_{C_b^2(\mathbb{R})} \sup_{t_n-T\leq s\leq r\leq t_n} |B_k(r)-B_k(s)|\Big\} \leq \varepsilon. \tag{6}$$

In the Brownian case, this property is easy to prove thanks to the independence of the increments and the classical Borel–Cantelli lemma. In the framework of the fBm, the increments are no longer independent and we will naturally employ a conditional version of the Borel–Cantelli lemma to prove this path property of the fBm. We will additionally make use of the Garsia–Rodemich–Rumsey inequality and Talagrand's small ball estimate (see the proof given in Section 5).

Despite these difficulties, one can state the following results concerning the invariant measure for the stochastic scalar conservation law with fractional forcing. Let us introduce the precise formulation of the result.

We denote as $\mathbb{D}$ the Skorohod space consisting of functions from $\mathbb{R}$ to $\mathbb{R}$ having discontinuities of the first kind. It is endowed with the metric

$$d(f,g) = \sum_{n\geq 1} 2^{-n}\big(1\vee d_n(f,g)\big)$$

where d_n is the usual distance of Skorohod on $[-n,n]$. Hence $(\mathbb{D},\mathcal{D})$ is a measurable space with $\mathcal{D}$ the sigma-algebra of Borel sets on $\mathbb{D}$.

In order to construct an invariant measure, we will construct an invariant solution. To this end we show that for almost all ω, there exists a solution $(t,x)\mapsto u^\sharp(t,x,\omega)$ starting from $u_0\equiv 0$ at $t_0=-\infty$. This solution will be built via minimizers of the action $\mathcal{A}_{t_0,0}$ when $t_0\to-\infty$ (see Section 4).

More precisely we will prove that there exists $u^\sharp$ from $\mathbb{R}\times\mathbb{R}\times\Omega$ to $\mathbb{R}$ such that:

(i) almost surely, $u^\sharp(t,\cdot,\omega)\in L^\infty(\mathbb{R})$ for any t;
(ii) almost surely, $u^\sharp(t,\cdot,\omega)\in\mathbb{D}$ for any t;
(iii) given t, the mapping $\omega\mapsto u^\sharp(t,\cdot,\omega)$ is measurable from $(\Omega,\mathcal{F})$ to $(\mathbb{D},\mathcal{D})$;
(iv) on any finite time interval $[t_1,t_2]$, for almost all ω, $(t,x)\mapsto u^\sharp(t,x,\omega)$ is a weak solution of (1) with initial data $u_0(x)=u^\sharp(t_1,x,\omega)$.

For the canonical space $\Omega=C_0(\mathbb{R},\mathbb{R})$, the space of continuous functions vanishing at 0, we denote as θ^τ the shift operator on Ω with increment τ defined by $\theta^\tau(\omega)(\cdot)=\omega(\cdot+\tau)-\omega(\tau)$ for any $\omega\in\Omega$. We stress the fact that the expression for the shift is modified compared to that of [20] because the fBm has stationary but not independent increments. This is the expression for the shift that leaves the two-sided fractional Brownian Wiener measure invariant. The solution operator S_ω^τ is defined for $v\in L^\infty(\mathbb{R})$ by $S_\omega^\tau(v)$, as the solution of (1) at time τ, with initial condition v at time $t_0=0$ when the realization of the noise is ω.

We have the following theorem.

Theorem 3. *We assume Hypotheses I and III. On $(\Omega\times\mathbb{D};\mathcal{F}\otimes\mathcal{D})$, the measure μ defined by*

$$\mu(d\omega,dv)=\mathbb{P}(d\omega)\delta_{u^\sharp(0,\cdot,\omega)}(dv) \tag{7}$$

is the unique measure that leaves invariant the (skew-product) transformation

$$\Omega\times\mathbb{D}\longrightarrow\Omega\times\mathbb{D}$$
$$(\omega,v)\longrightarrow(\theta^t\omega,S_\omega^t(v))$$

with given projection $\mathbb{P}$ on $(\Omega,\mathcal{F})$.

The proof of this result is given at the end of Section 4.

3. The dynamic programming equation

First we motivate the use of a variational principle, considering the one-dimensional (inviscid) Burgers equation

$$\partial_t u + \partial_x \left(\frac{u^2}{2}\right) = \frac{\partial}{\partial x} G(t,x) \quad t > 0,\ x \in \mathbb{R}$$

for an initial condition u_0 having discontinuities of the first kind (i.e. u_0 belongs to the Skorohod space $\mathbb{D}$). It is well known that there exists a unique entropy weak solution u given by

$$u(t,x) = \frac{\partial}{\partial_x}\left(\inf_{\substack{\xi\in C^1(0,t)\\ \xi(t)=x}}\left\{\mathcal{A}_{0,t} + \int_0^{\xi(0)} u_0(z)dz\right\}\right),$$

where

$$\mathcal{A}_{0,t}(\xi) = \int_0^t \left(\frac{1}{2}\dot{\xi}(s)^2 + G(t,\xi(s))\right) ds. \tag{8}$$

For two times t_1, t_2, we have denoted as $C^1(t_1,t_2)$ the space of continuously differentiable functions from $[t_1,t_2]$ to $\mathbb{R}$. This relation between Burgers' equation and the minimization problem is known as the Lax–Oleĭnik formula (see [13,15]) and the Hopf–Lax formula in its original context of Hamilton–Jacobi equations.

In the above equation we have intuitively assumed that G is a deterministic regular force. Now the source term in the action $\mathcal{A}_{\tau,t}$ is $\int_\tau^t \sum_{k\geq 1} F_k(\xi(s))dB_k(s)$ where the above integral is not a stochastic integral but a pathwise integral. Indeed, since the trajectories $\omega \to B_k(t)(\omega)$ are λ-Hölder continuous and ξ is differentiable, $\int_\tau^t F_k(\xi(s))dB_k(s)$ exists as a Riemann–Stieltjes integral thanks to a result of Young [21]. With $g(\cdot) := F_k(\xi(\cdot))$ one has

$$\int_\tau^t g(s)dB_k(s) = \lim_{\Delta\to 0}\sum_{i=0}^n g(t_i)(B_k(t_{i+1}) - B_k(t_i))$$

where the convergence holds uniformly in all finite partitions $\mathcal{P}_\Delta := \{\tau = t_0 \leq t_1 \leq \cdots t_{n+1} = t\}$ with $\max_i |t_{i+1} - t_i| < \Delta$. We define $\bar{B}(s) := B_k(s) - B(\tau)$ and we write

$$\begin{aligned}
\sum_{i=0}^n g(t_i)(B_k(t_{i+1}) - B_k(t_i)) &= \sum_{i=0}^n g(t_i)(\bar{B}(t_{i+1}) - \bar{B}(t_i))\\
&= -\sum_{i=0}^n \bar{B}(t_{i+1})\big(g(t_{i+1}) - g(t_i)\big)\\
&\quad + \sum_{i=0}^n \big\{\bar{B}(t_i)\big(g(t_{i+1}) - g(t_i)\big)\\
&\quad + \big(\bar{B}(t_{i+1}) - \bar{B}(t_i)\big)g(t_{i+1})\big\}\\
&= -\sum_{i=0}^n \bar{B}(t_{i+1})\big(g(t_{i+1}) - g(t_i)\big) + \bar{B}(t)g(t) - \bar{B}(\tau)g(\tau).
\end{aligned}$$

Consequently

$$\int_\tau^t g(s)dB_k(s) = -\int_t^\tau \big(B_k(s) - B_k(\tau)\big)\dot{g}(s)ds + \big(B_k(t) - B_k(\tau)\big)g(t)$$

and we rewrite the stochastic term of the action as

$$\int_\tau^t \sum_{k\geq 1} F_k(\xi(s))dB_k(s) = -\int_\tau^t \sum_{k\geq 1}\big(B_k(s)-B_k(\tau)\big)F_k'(\xi(s))\dot{\xi}(s)ds + \sum_{k\geq 1}\big(B_k(t)-B_k(\tau)\big)F_k(\xi(t)). \tag{9}$$

If $\xi(t)$ is fixed as x, then the second term in (9) is independent of ξ; hence as in [20] the action is redefined as for $\xi\in C^1(\tau,t)$ as

$$\mathcal{A}_{\tau,t}(\xi) = \int_\tau^t \left(\frac{1}{2}\dot{\xi}(s)^2 - \sum_{k\geq 1}\big(B_k(s)-B_k(\tau)\big)F_k'(\xi(s))\dot{\xi}(s)\right)ds + \sum_{k\geq 1}\big(B_k(t)-B_k(\tau)\big)F_k(\xi(t)).$$

Since the action is defined pathwise it depends on ω and hence should be denoted as $\mathcal{A}^\omega_{\tau,t}$. We will not do this for brevity of notation.

We stress the fact that (9) is a true integration by parts that allows us to rewrite the stochastic term and not a formal one, as was mentioned in [20].

In order to introduce the optimization problem, one can make a kind of change of variable in the variational formulation (2) and introduce a Hamilton–Jacobi–Bellman equation (HJB equation for short). Thus it is well known that these partial differential equations are related to a variational principle. Let us develop the following non-rigorous arguments. Let φ be a test function in $C_c^2(\mathbb{R}\times\mathbb{R})$; by an integration by parts one rewrites (2) as

$$\int_{t_0}^\infty\int_{\mathbb{R}} \partial_t\varphi(t,x)u(t,x)dxdt + \int_{t_0}^\infty\int_{\mathbb{R}} \partial_x\varphi(t,x)\,\Psi\big(u(t,x)\big)dxdt = -\int_{\mathbb{R}} u_0(x)\varphi(t_0,x)dx + \int_{\mathbb{R}}\int_{t_0}^\infty \partial_t\varphi(t,x)v(t,x)dtdx \tag{10}$$

with

$$v(t,x) = \sum_{k=1}^\infty F_k'(x)(B_k(t)-B_k(t_0)). \tag{11}$$

Consequently

$$\int_{t_0}^\infty\int_{\mathbb{R}} \partial_t\varphi(t,x)\big[u(t,x)-v(t,x)\big]dxdt + \int_{t_0}^\infty\int_{\mathbb{R}} \partial_x\varphi(t,x)\,\Psi\big(u(t,x)\big)dxdt = -\int_{\mathbb{R}} u_0(x)\varphi(t_0,x)dx$$

and with $w=u+v$ we obtain

$$\int_{t_0}^\infty\int_{\mathbb{R}} \partial_t\varphi(t,x)w(t,x)dxdt + \int_{t_0}^\infty\int_{\mathbb{R}} \partial_x\varphi(t,x)\,\Psi\big(w(t,x)+v(t,x)\big)dxdt = -\int_{\mathbb{R}} u_0(x)\varphi(t_0,x)dx.$$

Hence w is a solution of the stochastic scalar conservation law

$$\partial_t w + \mathrm{div}_x\,\Psi(w+v) = 0$$

and if we integrate this equation with respect to the space variable x, we derive the HJB equation

$$\partial_t W + \Psi(\partial_x W + v) = 0,$$

where W is such that $\partial_x W = w$. This HJB is related to an optimization problem with an action involving the Legendre transform of $p \mapsto \Psi(p+v)$. Thanks to the behavior under translation of the Legendre transformation, we have $\big(\Psi(\cdot + v)\big)^*(q) = \Psi^*(q) - vq$ and we deduce the expression of the action proposed in (5).

The above remarks are now made rigorous. First we express the action $\mathcal{A}_{t_0,t}$ as

$$\mathcal{A}_{t_0,t}(\xi) = \underbrace{\int_{t_0}^{t} L(s,\xi(s),\dot{\xi}(s))ds}_{\widetilde{\mathcal{A}}_{t_0,t}(\xi)} + V(t,\xi(t)) \quad \text{with}$$

$$\begin{aligned} L(s,x,p) &= \big(\Psi(\cdot + v(s,x))\big)^*(p) \\ &= \Psi^*(p) - \sum_{k\geq 1}\big(B_k(s) - B_k(t_0)\big)F_k'(x) \times p \quad \text{and} \\ V(t,x) &= \sum_{k\geq 1}\big(B_k(t) - B_k(t_0)\big)F_k(x). \end{aligned}$$

With U_0 such that $\partial_x U_0 = u_0$, we define

$$W(t,x) = \inf_{\substack{\xi\in H^1(t_0,t)\\ \xi(t)=x}} \left\{\widetilde{\mathcal{A}}_{t_0,t}(\xi) + U_0(\xi(t_0))\right\}. \tag{12}$$

We remark that $U_0(\xi(t_0)) = \int_0^{\xi(t_0)} u_0(z)dz$ and

$$W(t,x) = \inf_{\substack{\xi\in H^1(t_0,t)\\ \xi(t)=x}} \Big\{\mathcal{A}_{t_0,t}(\xi)\Big\} - V(t,x).$$

With regard to the classical calculus of variations, the left end point is fixed and the Hamiltonian is the Legendre transform of $p \mapsto L(t,x,p)$. Since we do not know of any precise reference where these changes are discussed, we briefly prove that there exists a minimizer of the action $\widetilde{A}_{t_0,t}$. We recall a definition:

Definition 3. We say that on the interval $[t_1,t_2]$, $\xi \in H^1(t_1,t_2)$ is a minimizer of the action $\widetilde{\mathcal{A}}_{t_1,t_2}$ if for any $\gamma \in H^1(t_1,t_2)$ with $\gamma(t_1) = \xi(t_1)$ and $\gamma(t_2) = \xi(t_2)$ we have $\widetilde{\mathcal{A}}_{t_1,t_2}(\xi) \leq \widetilde{\mathcal{A}}_{t_1,t_2}(\gamma)$.

We prove in the following proposition that the function W solves a Hamilton–Jacobi–Bellman equation and is semi-concave.

Proposition 4. *The function $(t,x) \mapsto W(t,x)$ is Lipschitz continuous and satisfies for almost all t, x the Hamilton–Jacobi–Bellman equation*

$$\partial_t W(t,x) + \Psi\left(\partial_x W(t,x) + \sum_{k\geq 1} F_k'(x)(B_k(t) - B_k(t_0))\right) = 0. \tag{13}$$

Moreover, for any t, the function $x \mapsto W(t,x)$ is semi-concave: there exists a constant K such that $x \mapsto W(t,x) - K(1 - \frac{1}{t-t_0})x^2$ is concave.

This proposition is proved in Appendix A.

Proof of Theorem 1. Now we prove the existence and uniqueness of the solution of (1).

Existence: Our candidate is $u = \partial_x W + v$ with W defined in (12) and v defined by (11). It is clearly adapted. Hypothesis I implies that $v(t,\cdot) \in L^\infty(\mathbb{R})$ and the Lipschitz property (34) for W implies that (ii) in Definition 1 holds true.

We prove the variational formulation. Let φ be a test function in $C_c^2(\mathbb{R}\times\mathbb{R})$. We integrate the HJB equation (13) against $\partial_x\varphi$ and we integrate by parts in order to obtain

$$\begin{aligned}
&-\int_{t_0}^\infty \int_{\mathbb{R}} \Psi\big(\partial_x W(t,x) + v(t,x)\big)\partial_x\varphi(t,x)dxdt\\
&\quad= \int_{t_0}^\infty \int_{\mathbb{R}} \partial_s W(t,x)\partial_x\varphi(t,x)dxdt\\
&\quad= -\int_{\mathbb{R}} W_0(x)\partial_x\varphi(t_0,x)dx + \int_{t_0}^\infty \int_{\mathbb{R}} \partial_x W(t,x)\partial_t\varphi(t,x)dxdt.
\end{aligned}$$

We have $\partial_x W_0(x) = \partial_x W(t_0,x) = u(t_0,x) + v(t_0,x) = u_0(x)$. By another integration by parts one obtains (10), which is equivalent to (2).

The entropy condition (3) is a consequence of the semi-concavity of W (see Proposition 4). Indeed, the concavity of $x \mapsto W(t,x) - Kx^2$ implies that its derivative is a decreasing function. Then for any $z > 0$,

$$\partial_x W(t,x+z) - 2K(x+z) \le \partial_x W(t,x) - 2Kx.$$

Moreover it holds that $\|\partial_x v(t,.)\|_\infty \le (t-t_0)^\lambda \sum_{k\ge1} \|F_k''\|_\infty \|B_k\|_{t,t_0,\lambda} := C$ and consequently $x \mapsto v(t,x) - 2Cx$ is a decreasing function and for any $z > 0$,

$$v(t,x+z) - 2C(x+z) \le v(t,x) - 2Cx.$$

The above two inequalities imply that $u = \partial_x W + v$ satisfies Oleĭnik's entropy condition (3).

Uniqueness: Since the random force in Eq. (1) does not depend on u, the uniqueness is given by classical arguments as in Theorem 3 in [5]. □

4. Action minimizers and generalized characteristics

In order to prove that there exists an invariant measure for the stochastic scalar conservation law (1), we will construct an invariant solution. For that purpose, we use minimizers of the action $\mathcal{A}_{\tau,t}$ which is defined for a piecewise regular curve ξ with $\xi(t) = x$ as

$$\begin{aligned}
\mathcal{A}_{\tau,t}(\xi) = \int_\tau^t &\Psi^*(\dot\xi(s)) - \sum_{k\ge1}(B_k(s) - B_k(\tau))F_k'(\xi(s))\dot\xi(s)ds\\
&+ \sum_{k\ge1}(B_k(t) - B_k(\tau))F_k(\xi(t)).
\end{aligned}$$

Using (9), the action can be expressed as

$$\mathcal{A}_{s,t}(\eta) = \int_s^t \Psi^*(\dot\eta(r))dr + \int_s^t \sum_{k\ge1} F_k(\eta(r))dB_k(r)$$

for any path $\eta \in C^1(s,t)$. A fundamental object is the one-sided minimizer defined as follows.

Definition 4. Let $t \in \mathbb{R}$. A piecewise C^1 curve $\xi :]-\infty, t] \to \mathbb{R}$ is a one-sided minimizer if

(i) for any $\tilde{\xi} \in H^1(-\infty, t)$ such that $\tilde{\xi}(t) = \xi(t)$ and $\tilde{\xi} = \xi$ on $]-\infty, \tau]$ for some $\tau < t$, it holds that $\mathcal{A}_{s,t}(\xi) \le \mathcal{A}_{s,t}(\tilde{\xi})$ for any $s \le \tau$;
(ii) for any $s \le t$, $|\xi(s) - \xi(t)| \le 1$.

Most of the properties of these one-sided minimizers are quite basic facts proved in [20]. Nevertheless we will make them precise because we work with a general convex flux instead of the square function that was used in Burgers' case. We stress the fact that we choose a slightly different definition of the one-sided minimizer (we impose the boundedness when the value $\xi(t)$ is fixed) because we do not work on the torus as in [20] but on $\mathbb{R}$.

Euler–Lagrange equations and properties of the action minimizers

The Euler–Lagrange equation can be formally deduced from the following classical computation. If we want to find two curves γ and v such that $v(t) = u(t, \gamma(t))$, then we have the relation

$$dv(t) = \partial_t u(t, \gamma(t)) + \partial_x u(t, \gamma(t))\dot{\gamma}(t).$$

With $\dot{\gamma}(t) = \Psi'(u(t, \gamma(t)))$ (or equivalently $v(t) = (\Psi')^{-1}(\dot{\gamma}(t))$), together with (1), one writes

$$dv(t) = \partial_t u(t, \gamma(t)) + \partial_x \Psi(u(t, \gamma(t))),$$

and we obtain the Euler–Lagrange equation

$$\begin{cases} \dot{\gamma}(s) = \Psi'(v(s)) \\ dv(s) = \sum_{k\ge 1} F_k'(\gamma(\tau)) dB_k(\tau). \end{cases} \tag{14}$$

The curve γ is a generalized characteristic in the sense of Dafermos (see [1]). Eq. (14) is a generalization of the Euler–Lagrange equation (2.3) in [20] obtained for $\Psi(z) = z^2/2$:

$$\begin{cases} \dot{\gamma}(s) = v(s) \\ dv(s) = \sum_{k\ge 1} F_k'(\gamma(\tau)) dB_k(\tau). \end{cases}$$

In the following proposition, we prove that:

- there exists effectively a unique solution to Eq. (14),
- a minimizer of the action satisfies an Euler–Lagrange equation and is a regular curve,
- we give estimation for the velocities of such a minimizer.

For any times t_1, t_2 and any $x_1, x_2 \in \mathbb{R}$, we define

$$\mathcal{H}_{x_1,x_2}^{t_1,t_2} = \left\{ \xi \in H^1(t_1, t_2); \xi(t_1) = x_1, \xi(t_2) = x_2 \right\}.$$

Proposition 5. *Let two times t_1 and t_2 be fixed.*

(a) *For ξ_2 and v_2 two fixed real numbers, there exists a unique solution $\xi \in C^1(t_1, t_2)$ to the Euler–Lagrange equation*

$$\dot{\xi}(s) = \Psi'(v(s))$$
$$v(s) = v(t_2) + \int_s^{t_2} \sum_{k\geq 1} F_k'(\xi(r))dB_k(r) \quad t_1 \leq s \leq t_2 \tag{15}$$

with the terminal condition $(\xi(t_2), \dot{\xi}(t_2)) = (\xi_2, \Psi'(v_2))$.

(b) *If* γ *is a minimizer of* $\mathcal{A}$ *on* $[t_1, t_2]$, *that is*

$$\mathcal{A}_{t_1,t_2}(\gamma) = \inf_{\xi \in \mathcal{H}^{t_1,t_2}_{x_1,x_2}} \left\{ \int_{t_1}^{t_2} \Psi^*(\dot{\xi}(s)) - \sum_{k\geq 1}(B_k(s) - B_k(t_1)) F_k'(\xi(s))\dot{\xi}(s)ds \right.$$
$$\left. + \sum_{k\geq 1}(B_k(t_2) - B_k(t_1)) F_k(\xi(t_2)) \right\},$$

then $\dot{\gamma} \in C^1(t_1, t_2)$ *satisfies for* $t_1 \leq r \leq s \leq t_2$

$$(\Psi^*)'(\dot{\gamma}(s)) - (\Psi^*)'(\dot{\gamma}(r)) = \int_r^s \sum_{k\geq 1} F_k'(\gamma(\tau))dB_k(\tau). \tag{16}$$

(c) *If* γ *is a minimizer of the action* $\mathcal{A}$ *on the time interval* $[t_1, t_2]$ *with* $\gamma(t_1) = x_1, \gamma(t_2) = x_2$ *and* $t_2 - t_1 \geq 1$, *then there exists a constant c such that*

$$\|\dot{\gamma}\|_{t_1,t_2,\infty} \leq c\, C_{t_1,t_2} + \left((t_2 - t_1)^{\frac{-1}{1+\alpha}} + C_{t_1,t_2}(t_2 - t_1)^{\frac{\alpha}{1+\alpha}}\right)(x_2 - x_1)^{\frac{1+\beta}{1+\alpha}}$$
$$+ \left((t_2 - t_1)^{\frac{-1}{1+\alpha}} + C_{t_1,t_2}(t_2 - t_1)^{\frac{\alpha}{1+\alpha}}\right)\left(C_{t_1,t_2}^{\frac{1}{1+\alpha}} + C_{t_1,t_2}^{\frac{1}{\alpha}}(t_2 - t_1)^{\frac{1}{1+\alpha}}\right) \tag{17}$$

with $C_{t_1,t_2} = \sum_{k\geq 1} \|F_k\|_{C^2} \left\{\sup_{t_1 \leq r \leq r' \leq t_2} |B_k(r) - B_k(r')|\right\}$.

We recall that we work on each trajectory of the random force. The proof of these results is postponed to Appendix B.

Existence and uniqueness of one-sided minimizers

The following proposition establishes the existence of a one-sided minimizer. It is a short rewriting of the one contained in [20], which takes care of the fact that we do not work on the torus.

Proposition 6. *For every* $x \in \mathbb{R}$ *and* $t \in \mathbb{R}$, *there exists a one-sided minimizer* γ *such that* $\gamma(t) = x$.

Proof. Let n be an integer such that $-n < t$ and γ_n a minimizer of $\mathcal{A}_{-n,t}$ satisfying $\gamma_n(t) = x$, $\gamma_n(-n) = x + 1$ and $\sup_{-n\leq s\leq t} |\gamma_n(s) - x| \leq 1$. From the proof of Proposition 4, such a γ_n exists. For $-n < s < t$ we have $\|\dot{\gamma}_n\|_{s,t,\infty} \leq K$ by (17), where K depends on s and t but does not depend on x. Hence, up to a subsequence, there exists $\gamma \in H^1(s, t)$ such that $\lim_{n\to\infty} \gamma_n = \gamma$ in $C(s, t)$ and $\lim_{n\to\infty} \dot{\gamma}_n = \dot{\gamma}$ weakly in $L^2(s, t)$. From the Euler–Lagrange equation (15) it follows that $\lim_{n\to\infty} \gamma_n = \gamma$ in $C^1(s, t)$ (after a new extraction of a subsequence). A diagonal process implies that there exists $\gamma \in C^1(-\infty, t)$ such that $\lim_{n\to\infty} \gamma_n = \gamma$ for the C^1 convergence on any compact of $]-\infty, t]$.

It remains to prove that γ is a one-sided minimizer. By construction, (ii) in Definition 4 is satisfied. Let a curve $\xi \in H^1(-\infty, t)$ with $\xi(t) = x$ and $\xi = \gamma$ on $]-\infty, \tau]$ for some τ. Without loss of generality we can take $\xi \in C^1(-\infty, t)$ because the action can be strictly decreased by smoothing a curve containing corners (see Fact 2 page 885 in [20]). Fix $s \leq \tau$ and let $(\xi_n)_{n\geq 1}$ be a sequence in $C^1(s,t)$ such that $\xi_n(s) = \gamma_n(s)$, $\xi_n(t) = x$ and $\lim_{n\to\infty} \xi_n = \xi$ in $C^1(s,t)$. We have $\lim_{n\to\infty} \xi_n(s) = \lim_{n\to\infty} \gamma_n(s) = \gamma(s) = \xi(s)$. Using Hypothesis II(d) we obtain

$$\begin{aligned}
|\mathcal{A}_{s,t}(\xi) - \mathcal{A}_{s,t}(\xi_n)| \leq & \int_s^t |\Psi^*(\dot{\xi}(r)) - \Psi^*(\dot{\xi}_n(r))| dr \\
& + \int_s^t \left| \sum_{k\geq 1} (B_k(r) - B_k(s)) F_k'(\xi_n(r)) \big(\dot{\xi}(r) - \dot{\xi}_n(r)\big) \right| dr \\
& + \int_s^t \left| \sum_{k\geq 1} (B_k(r) - B_k(s)) \big(F_k'(\xi(r)) - F_k'(\xi_n(r))\big) \dot{\xi}(r) \right| dr \\
& + \left| \sum_{k\geq 1} (B_k(t) - B_k(s)) \big(F_k(\xi_n(t)) - F_k(\xi(t))\big) \right| \\
\leq & \ (C(R) + C_{s,t}) \|\dot{\xi} - \dot{\xi}_n\|_{L^1(s,t)} + C_{s,t} \|\dot{\xi}\|_{L^2(s,t)} \|\xi - \xi_n\|_{L^2(s,t)}
\end{aligned}$$

with $C_{s,t} = \sum_{k\geq 1} \|F_k\|_{C^2} \left\{ \sup_{t_1 \leq r \leq r' \leq t_2} |B_k(r) - B_k(r')| \right\}$ defined as in (17) and R such that $\|\dot{\xi}\|_{s,t,\infty} \vee \left(\sup_{n\geq 1} \|\dot{\xi}_n\|_{s,t,\infty} \right) \leq R$. The above estimation implies that $\lim_{n\to\infty} \mathcal{A}_{s,t}(\xi_n) = \mathcal{A}_{s,t}(\xi)$. Moreover,

$$\left| \mathcal{A}_{s,t}(\gamma) - \mathcal{A}_{s,t}(\gamma_n) \right| \leq C \|\gamma - \gamma_n\|_{C^1(s,t)} \xrightarrow[n\to\infty]{} 0,$$

with C depending on $\|\gamma\|_{C^1(s,t)}$ and for $-n \leq s$, $\mathcal{A}_{s,t}(\gamma_n) \leq \mathcal{A}_{s,t}(\xi_n)$ because γ_n is a minimizer of $\mathcal{A}_{-n,t}$. Therefore

$$\mathcal{A}_{s,t}(\gamma) = \lim_{n\to\infty} \mathcal{A}_{s,t}(\gamma_n) \leq \lim_{n\to\infty} \mathcal{A}_{s,t}(\xi_n) = \mathcal{A}_{s,t}(\xi).$$

We conclude that γ is a one-sided minimizer. $\square$

The intersection of one-sided minimizers

We will use for the first time the randomness of the force. Theorem 2 states that the fractional Brownian noise is arbitrarily small on an infinite number of arbitrarily long time intervals: for all $\varepsilon > 0$, $T > 0$, for almost all ω, there exists a sequence of random times $(t_n(\omega))_{n\geq 1}$ such that $t_n(\omega) \to -\infty$ and

$$\forall n, \quad C_{t_n - T, t_n} = \sup_{t_n - T \leq s \leq t_n} \sum_{k\geq 1} \left\{ \|F_k\|_{C_b^2(\mathbb{R})} |B_k(s) - B_k(t_n)| \right\} \leq \varepsilon.$$

We remark that this property of the noise implies that the velocity of a minimizer will be as small as we want, by (17).

The following proposition states that two different one-sided minimizers with the same ends cannot intersect each other more than once (see [20, Lemma 3.2]). So if two one-sided minimizers intersect more than once, they coincide on their common interval of definition.

Proposition 7. *For almost all ω, for any distinct one-sided minimizers γ_1 and γ_2 on $]-\infty, t_1]$ and $]-\infty, t_2]$ respectively, the following result holds. Assume that γ_1 and γ_2 intersect at time t in a point x; then $t_1 = t_2 = t$ and $\gamma_1(t_1) = \gamma_2(t_2) = x$.*

The proof of this result is exactly the same as the proof of Lemma 3.2 in [20] so we do not repeat it. Nevertheless, it is based on [20, Lemma 3.3] which we do recall and briefly prove because there are minor modifications due to our fractional noise.

Lemma 8. *Almost surely, for any $\varepsilon > 0$ and any two one-sided minimizers $\gamma_1 \in C^1(-\infty, t_1)$ and $\gamma_2 \in C^1(-\infty, t_2)$, there exists $T = T(\varepsilon)$ and a sequence of random times $t_n = t_n(\omega, \varepsilon) \to -\infty$ such that*

$$\left|\mathcal{A}_{t_n-T,t_n}(\gamma_i) - \mathcal{A}_{t_n-T,t_n}(\zeta)\right| < \varepsilon, \quad \textit{for } i = 1, 2 \textit{ and } \zeta \in \{\widehat{\gamma_1\gamma_2}, \widehat{\gamma_2\gamma_1}\}$$

where $\widehat{\gamma_1\gamma_2}$ and $\widehat{\gamma_2\gamma_1}$ are reconnecting curves defined by

$$\widehat{\gamma_1\gamma_2}(s) = \frac{t_n - s}{T}\gamma_1(s) - \frac{t_n - T - s}{T}\gamma_2(s)$$
$$\widehat{\gamma_2\gamma_1}(s) = \frac{t_n - s}{T}\gamma_2(s) - \frac{t_n - T - s}{T}\gamma_1(s).$$

Proof. For T sufficiently large, we use (6) in order to find a sequence of random times $(t_n)_{n\ge1}$ such that $\lim_{n\to\infty} t_n = -\infty$ and

$$\forall n, \quad C_{t_n-T,t_n} = \sup_{t_n-T\le s\le t_n} \sum_{k\ge1}\left\{\|F_k\|_{C_b^2(\mathbb{R})}|B_k(s) - B_k(t_n)|\right\} \le \frac{1}{T}, \tag{18}$$

where the notation C_{t_n-T,t_n} comes from (17).

Now we make the following remark. If a curve γ minimizes the action on the interval $[s, t]$, then for any $s < r < t$, its restriction on $[s, r]$ will minimize the action with respect to curves in $H^1(s, r)$ having the same ends as γ at s and r. Indeed suppose that there is a minimizer $\xi \neq \gamma$ on $[s, r]$ such that $\mathcal{A}_{s,r}(\xi) = \mathcal{A}_{s,r}(\gamma) - \varepsilon$. Using (9), the action can be written using a true pathwise integral with respect to the noise, so the action of any path $\eta \in C^1(s, t)$ is expressed as

$$\mathcal{A}_{s,t}(\eta) = \int_s^t \Psi^*(\dot\eta(s))ds + \int_s^t \sum_{k\ge1} F_k(\eta(s))dB_k(s).$$

Hence the action is additive with respect to C^1 curves ($\mathcal{A}_{r,t}(\eta) = \mathcal{A}_{r,s}(\eta) + \mathcal{A}_{s,t}(\eta)$ if η is C^1). Considering the curve $\widehat{\xi\gamma_{r,t}}$ obtained by gluing the path ξ to the restriction of γ on $[r, t]$, we observe that

$$\mathcal{A}_{s,t}(\widehat{\xi\gamma_{r,t}}) = \mathcal{A}_{s,r}(\gamma) - \varepsilon + \mathcal{A}_{r,t}(\gamma) = \mathcal{A}_{s,t}(\gamma) - \varepsilon < \mathcal{A}_{s,t}(\gamma)$$

which contradicts the fact that γ is a minimizer on $[s, t]$.

Therefore, the one-sided minimizers γ_i are minimizers on each time interval $[t_n - T, t_n]$ and we use the inequalities (17) and (18) to obtain that for any n

$$\sup_{t_n-T\le s\le t_n} |\dot\gamma_i(s)| \le \frac{c}{T} + 2c\left(T^{-\frac{1}{1+\alpha}} + T^{-\frac{2}{1+\alpha}} + T^{-\frac{1}{\alpha}}\right)$$
$$\le \frac{\tilde c}{T^{1/(1+\alpha)}}.$$

Consequently, for $\zeta \in \{\widehat{\gamma_1\gamma_2}, \widehat{\gamma_2\gamma_1}\}$ we have $\|\dot{\zeta}\|_{t_n-T,t_n,\infty} \le \frac{2\tilde{c}}{T^{1/(1+\alpha)}}$. Using $|\Psi^*(v)| \le c_2|v|^{1+\beta}$ and (18) we have

$$
\begin{aligned}
|\mathcal{A}_{t_n-T,t_n}(\gamma_i) - \mathcal{A}_{t_n-T,t_n}(\zeta)| &\le \int_{t_n-T}^{t_n} \left|\Psi^*(\dot{\gamma}_i(s)) - \Psi^*(\dot{\zeta}(s))\right| ds \\
&+ \int_{t_n-T}^{t_n} \left|\sum_{k\ge 1}(B_k(s) - B_k(t_n-T))F_k'(\zeta)\big(\dot{\gamma}_i(s) - \dot{\zeta}(s)\big)\right| ds \\
&+ \int_{t_n-T}^{t_n} \left|\sum_{k\ge 1}(B_k(s) - B_k(t_n-T))\big(F_k'(\gamma_i(s)) - F_k'(\zeta(s))\big)\dot{\gamma}_i(s)\right| ds \\
&+ \left|\sum_{k\ge 1}(B_k(t_n) - B_k(t_n-T))\big(F_k(\zeta(t_n)) - F_k(\gamma_i(t_n))\big)\right| \\
&\le C\left(\frac{2\,T}{T^{\frac{1+\beta}{1+\alpha}}} + T\frac{4\,C_{t_n-T,t_n}}{T^{1/(1+\alpha)}} + 2\,C_{t_n-T,t_n}\right) \\
&\le C\left(\frac{2}{T^{\frac{\beta-\alpha}{1+\alpha}}} + \frac{4}{T^{1/(1+\alpha)}} + \frac{2}{T}\right)
\end{aligned}
\tag{19}
$$

where C is a numerical constant. The result follows by choosing T such that the right hand side of (19) is less than ε. □

An invariant measure: existence and uniqueness

In this subsection, we prove Theorem 3. First we construct the invariant solution $u^\sharp$. We denote as $\mathcal{M}_{t,x}$ the family of all one-sided minimizers with end x at time t. We define

$$u^\sharp(t,x,\omega) = \inf_{\gamma\in\mathcal{M}_{t,x}} \dot{\gamma}(t). \tag{20}$$

Proposition 7 implies an important property of one-sided minimizers. To any $x \in \mathbb{R}$ such that the cardinal of $\mathcal{M}_{t,x}$ is at least 2 (this means that more than one one-sided minimizer comes to x at time t), there corresponds a non-trivial segment $I(x) = [\gamma_1(t-T), \gamma_2(t-T)]$, where $\gamma_1 < \gamma_2$ on $]-\infty, t]$ because two different one-sided minimizers cannot intersect each other more than once. Thus the segments $I(x)$ are mutually disjoint. Consequently, for almost all ω, the set of $x \in \mathbb{R}$ with more than one one-sided minimizer coming to x at time t is at most countable. The above argument relies on the fact that an infimum is taken in (20) of $u^\sharp$ and this is the key point for proving that $u^\sharp(t,\cdot,\omega) \in \mathbb{D}$ (see [20, Lemma 3.8]).

The fact that $u^\sharp \in L^\infty(\mathbb{R})$ is a consequence of (17). The measurability issues can be treated as in [20, Lemma 3.9].

Moreover on any finite time interval $[t_1,t_2]$, for almost all ω, $(t,x) \mapsto u^\sharp(t,x,\omega)$ is a weak solution of (1) with initial data $u_0(x) = u^\sharp(t_1,x,\omega)$. This is obtained by construction of $u^\sharp$. Hence $S_\omega^t(u^\sharp(0,\cdot,\omega)) = u^\sharp(t,\cdot,\omega)$. Thus the measure μ defined in Theorem 3 leaves the skew-product transformation invariant.

It only remains to prove the uniqueness. This is also done in the proof of the Theorem 4.2 in [20]. Let us give a few details. For λ another invariant measure, we denote as λ_ω its projection on Ω in such a way that we may write that $\lambda(d\omega, dv) = \int_\Omega \lambda_\omega(dv)\mathbb{P}(d\omega)$. The invariance of λ implies that there exists a subset D of $\mathbb{D}$ such that $\lambda(D^c) = 0$ and with the property that for any

$v \in D$ and any $n \in \mathbb{N}$, there exists v_n such that $H_{-n}(v_n) = v$ where the operator H_t maps the solution of (1) at a negative time t to this solution at time 0. By repeating the end of the proof of Proposition 6, we have that if a solution of (1) can be extended to arbitrary negative times, this solution coincides with $u^\sharp$ at time $t = 0$ for almost all x (because the set of $x \in \mathbb{R}$ with more than one one-sided minimizer coming to x at time 0 is at most countable). Hence $v(x) = u^\sharp(0, x)$ almost everywhere and $\lambda_\omega(dv) = \delta_{u^\sharp(0,\cdot,\omega)}(dv)$, and we have the uniqueness.

5. An asymptotic property of fBm

The key point for proving the existence of an invariant measure was the fact that the fractional Brownian motion has periods of arbitrary length and arbitrarily small amplitude oscillation as time goes to $-\infty$. The result stated in Theorem 2 is recalled below.

Theorem 2: For all $\varepsilon > 0, T > 0$, for almost all ω, there exists a sequence of random times $(t_n(\omega))_{n\geq 1}$, such that $t_n(\omega) \to -\infty$ and

$$\forall n, \quad \sum_{k\geq 1} \left\{ \|F_k\|_{C_b^2(\mathbb{R})} \sup_{t_n - T \leq s \leq r \leq t_n} |B_k(r) - B_k(s)| \right\} \leq \varepsilon.$$

Before proving this theorem, we will recall and prove some basic facts concerning the fBm defined on the real line $\mathbb{R}$.

First we deal with the moving average representation of the fBm $(B(t))_{t\in\mathbb{R}}$. For $s, t \in \mathbb{R}$, we define

$$f_t(s) = c_H \left((t-s)_+^{H-\frac{1}{2}} - (-s)_+^{H-\frac{1}{2}} \right)$$

with

$$c_H = \left(\int_0^\infty \left((1+s)^{H-\frac{1}{2}} - s^{\frac{1}{2}} \right)^2 ds + \frac{1}{2H} \right)^{-\frac{1}{2}}.$$

Notice that $\int_\mathbb{R} f_t^2(s) ds < \infty$ and more precisely, if $H \neq 1/2$, $s \mapsto f_t(s)$ behaves like $(-s)^{H-3/2}$ when $s \to -\infty$ which is square integrable at $-\infty$. Thus the fBm can be written as

$$B(t) = \int_\mathbb{R} f_t(s) dW_s$$

where the process $(W_t)_{t\in\mathbb{R}}$ is a two-sided classical Brownian motion which is obtained by gluing two independent copies of one-sided Brownian motions together at time $t = 0$.

Since we are interested in the oscillations of the fBm, we express its increments for $t < t' < 0$ as

$$\begin{aligned} B(t) - B(t') &= \int_\mathbb{R} c_H \left\{ (t-r)_+^{H-\frac{1}{2}} - (t'-r)_+^{H-\frac{1}{2}} \right\} dW_r \\ &= \int_{-\infty}^t c_H \left\{ (t-r)^{H-\frac{1}{2}} - (t'-r)^{H-\frac{1}{2}} \right\} dW_r + \int_t^{t'} c_H (t'-r)^{H-\frac{1}{2}} dW_r \\ &= \int_\mathbb{R} g_{t,t'}(r) dW_r \end{aligned}$$

where

$$g_{t,t'}(r) = c_H \left\{ (t-r)^{H-\frac{1}{2}} - (t'-r)^{H-\frac{1}{2}} \right\} \mathbf{1}_{]-\infty,t]}(r) + c_H (t'-r)^{H-\frac{1}{2}} \mathbf{1}_{[t,t']}(r).$$

Let $\mathcal{F}_s$ be the sigma-algebra generated by the family of random variables $\{B(r); -\infty < r \le s\}$. We remark that for $s \le 0$, $\mathcal{F}_s \subseteq \sigma\{W_r; -\infty < r \le s\} := \mathcal{F}^W_{-\infty,s}$. Then we deduce the following expression: for any $-\infty < s \le t \le t' \le 0$,

$$\mathbb{E}(B(t) - B(t')|\mathcal{F}_s) = \mathbb{E}\left[\int_{-\infty}^{s} c_H \left\{(t-r)^{H-\frac{1}{2}} - (t'-r)^{H-\frac{1}{2}}\right\} dW_r \middle| \mathcal{F}_s\right]. \tag{21}$$

The proof of Theorem 2 is based on the following reversed conditional Borel–Cantelli lemma.

Lemma 9. *Let $(\mathcal{F}_n)_{n\ge1}$ be a decreasing sequence of σ-fields and $(A_n)_{n\ge1}$ a sequence of events such that $A_n \in \mathcal{F}_n$. Then the events*

$$\left\{\sum_{k\ge1} \mathbf{1}_{A_k} < \infty\right\} \quad \text{and} \quad \left\{\sum_{k\ge1} \mathbb{E}\left(1_{A_k}|\mathcal{F}_{k+1}\right) < \infty\right\}$$

are almost surely equal.

Proof. Let $M_n = \mathbf{1}_{A_n} - \mathbb{E}(\mathbf{1}_{A_n}|\mathcal{F}_{n+1})$. We have $\mathbb{E}(M_n|\mathcal{F}_{n+1}) = 0$ so $(M_n)_{n\ge1}$ is a reversed martingale difference sequence. This means that if we set for negative integers $\tilde{\mathcal{F}}_n = \mathcal{F}_{-n}$ and $\tilde{M}_n = M_{-n}$, $(\tilde{M}_n)_{n\le-1}$ is a martingale difference sequence with respect to the filtration $(\tilde{\mathcal{F}}_n)_{n\le-1}$. Moreover, $(\tilde{S}_n)_{n\le-1}$ defined by $\tilde{S}_n = \sum_{k=n}^{k=-1} \tilde{M}_k$ is an $(\tilde{\mathcal{F}}_n)_{n\le-1}$ martingale.

Step 1: We prove that the following inclusion holds almost surely:

$$\left\{\sum_{k\ge1} \mathbb{E}\left(1_{A_k}|\mathcal{F}_{k+1}\right) < \infty\right\} \subset \left\{\sum_{k\ge1} \mathbf{1}_{A_k} < \infty\right\}. \tag{22}$$

Using the stopping times

$$\tau_K = \inf\left\{k \le -1; \sum_{-\infty}^{k=n+1} \mathbb{E}\left(\tilde{M}_n^2|\tilde{\mathcal{F}}_{n-1}\right) > K\right\} \quad \text{for } K > 0,$$

and following [17, Theorem 2.8.7] (see also Theorem 4.1(v), p. 320 in [4]) we obtain that

$$\left\{\sum_{k\le-1} \mathbb{E}(\tilde{M}_k^2|\tilde{\mathcal{F}}_{k-1}) < \infty\right\} \subseteq \left\{\sum_{k\le-1} \tilde{M}_k < \infty\right\} \quad \text{almost surely.}$$

In the reverse martingale formulation this means that

$$\left\{\sum_{k\ge1} \mathbb{E}(M_k^2|\mathcal{F}_{k+1}) < \infty\right\} \subseteq \left\{\sum_{k\ge1} M_k < \infty\right\} \quad \text{almost surely.} \tag{23}$$

Hence we write

$$\begin{aligned} \mathbb{E}(M_k^2|\mathcal{F}_{k+1}) &= \mathbb{E}(\mathbf{1}_{A_k}^2|\mathcal{F}_{k+1}) - \left(\mathbb{E}(\mathbf{1}_{A_k}|\mathcal{F}_{k+1})\right)^2 \\ &= \mathbb{E}(\mathbf{1}_{A_k}|\mathcal{F}_{k+1})\Big[1 - \mathbb{E}(\mathbf{1}_{A_k}|\mathcal{F}_{k+1})\Big] \\ &\le \mathbb{E}(\mathbf{1}_{A_k}|\mathcal{F}_{k+1}). \end{aligned} \tag{24}$$

If $\sum_{k\ge1} \mathbb{E}(\mathbf{1}_{A_k}|\mathcal{F}_{k+1}) < \infty$, then by (23) $\sum_{k\ge1} M_k$ is almost surely convergent. Using $\sum_{k\ge1} \mathbf{1}_{A_k} = \sum_{k\ge1} M_k + \sum_{k\ge1} \mathbb{E}(\mathbf{1}_{A_k}|\mathcal{F}_{k+1})$, we deduce (22).

Step 2: We prove that

$$\left\{\sum_{k\geq 1}\mathbb{E}\left(1_{A_k}|\mathcal{F}_{k+1}\right)=+\infty\right\}\subset\left\{\sum_{k\geq 1}\mathbf{1}_{A_k}=+\infty\right\}. \tag{25}$$

For the event

$$\left\{\sum_{k\geq 1}\mathbb{E}(\mathbf{1}_{A_k}|\mathcal{F}_{k+1})=+\infty\right\}\cap\left\{\sum_{k\geq 1}\mathbb{E}(M_k^2|\mathcal{F}_{k+1})<\infty\right\},$$

thanks to (23) we have $\sum_{k\geq 1}M_k<\infty$ and consequently

$$\sum_{k\geq 1}\mathbf{1}_{A_k}=\sum_{k\geq 1}M_k+\sum_{k\geq 1}\mathbb{E}(\mathbf{1}_{A_k}|\mathcal{F}_{k+1})=+\infty.$$

So it remains to prove the inclusion (25) for the event

$$A=\left\{\sum_{k\geq 1}\mathbb{E}(\mathbf{1}_{A_k}|\mathcal{F}_{k+1})=+\infty\right\}\cap\left\{\sum_{k\geq 1}\mathbb{E}(M_k^2|\mathcal{F}_{k+1})=+\infty\right\}.$$

If $(\tilde{H}_n)_{n\leq -1}$ is an $(\tilde{\mathcal{F}}_n)_{n\leq -1}$-martingale we define for $n\leq -1$

$$\langle\tilde{H}\rangle_n=\sum_{k=n}^{k=-1}\mathbb{E}\left[(\tilde{H}_k-\tilde{H}_{k-1})^2|\tilde{\mathcal{F}}_{k-1}\right].$$

Now let $(\tilde{X}_n)_{n\leq -1}$ be the martingale defined by

$$\tilde{X}_n=\sum_{k=n}^{k=-1}\frac{\tilde{S}_k-\tilde{S}_{k-1}}{1+\langle\tilde{S}\rangle_k^{3/4}}=\sum_{k=n}^{k=-1}\frac{\tilde{S}_k-\tilde{S}_{k-1}}{1+\left(\sum\limits_{j=k}^{j=-1}\mathbb{E}(\tilde{M}_j^2|\tilde{\mathcal{F}}_{j-1})\right)^{3/4}}.$$

We are working on the event A on which $\lim_{n\to-\infty}\langle\tilde{S}\rangle_n=+\infty$. Since

$$\begin{aligned}\langle\tilde{X}\rangle_n&=\sum_{k=n}^{k=-1}\frac{\langle\tilde{S}\rangle_k-\langle\tilde{S}\rangle_{k-1}}{\left(1+\langle\tilde{S}\rangle_k^{3/4}\right)^2}\\&\leq\sum_{-\infty}^{k=-1}\int_{\langle\tilde{S}\rangle_{k-1}}^{\langle\tilde{S}\rangle_k}\frac{dt}{\left(1+t^{3/4}\right)^2}\\&\leq\int_0^\infty\frac{dt}{\left(1+t^{3/4}\right)^2}<\infty,\end{aligned}$$

the martingale $(\tilde{X}_n)_{n\leq -1}$ converges almost surely on A. Kronecker's lemma implies that

$$\lim_{n\to-\infty}\frac{1}{1+\langle\tilde{S}\rangle_n^{3/4}}\sum_{k=n}^{k=-1}\left(\tilde{S}_k-\tilde{S}_{k-1}\right)=0,$$

and thus $\lim_{n\to-\infty}(\sum_{k=n}^{k=-1}\tilde{M}_k)/\langle\tilde{S}\rangle_n^{3/4}=0$ or equivalently

$$\lim_{n\to+\infty}\frac{\sum\limits_{k=1}^{n}M_k}{\left(\sum\limits_{k=1}^{n}\mathbb{E}(M_k^2|\mathcal{F}_{k+1})\right)^{3/4}}=0.$$

Then there exists a random $K>0$ such that for sufficiently large n we have $\sum_{k=1}^{n}M_k\geq -1/2\left(\sum_{k=1}^{n}\mathbb{E}(M_k^2|\mathcal{F}_{k+1})\right)^{3/4}$. With (24) we may write

$$\begin{aligned}\sum_{k=1}^{n}\mathbf{1}_{A_k}&=\sum_{k=1}^{n}M_k+\sum_{k=1}^{n}\mathbb{E}(\mathbf{1}_{A_k}|\mathcal{F}_{k+1})\\&\geq\sum_{k=1}^{n}M_k+\sum_{k=1}^{n}\mathbb{E}(M_k^2|\mathcal{F}_{k+1})\\&\geq\left(\sum_{k=1}^{n}\mathbb{E}(M_k^2|\mathcal{F}_{k+1})\right)^{3/4}\left[-1/2+\left(\sum_{k=1}^{n}\mathbb{E}(M_k^2|\mathcal{F}_{k+1})\right)^{1/4}\right],\end{aligned}$$

so $\lim_{n\to+\infty}\sum_{k=1}^{n}\mathbf{1}_{A_k}=+\infty$. The proof is now complete. □

Proof of Theorem 2. Let $\varepsilon>0$ and $T>0$ be fixed. Let $(t_n)_{n\geq1}$ be a decreasing sequence of negative real numbers such that

$$\begin{cases}\lim\limits_{n\to\infty}t_n=-\infty;\\ t_{n+1}<t_n-T\quad\text{and}\\ \sum\limits_{n\geq1}(t_n-t_{n+1})^{H-1}<\infty.\end{cases}$$

Step 1: We prove the property for a single fBm $(B(t))_{t\in\mathbb{R}}$. We define $\mathcal{F}_{t_n}=\sigma\{B(r);-\infty<r\leq t_n\}$ and for $t\geq t_{n+1}$ we set

$$\begin{aligned}&B^{n+1}(t)=\mathbb{E}(B(t)|\mathcal{F}_{t_{n+1}})\\&\overline{B}^{n+1}(t)=B(t)-B^{n+1}(t).\end{aligned}$$

By the Gaussian properties of the fBm it follows that $\overline{B}^{n+1}(t)$ is independent of $\mathcal{F}_{t_{n+1}}$. We set

$$\begin{aligned}A_n(\varepsilon)&=\left\{\sup_{t_n-T\leq t,s\leq t_n}|B(t)-B(s)|\leq\varepsilon\right\},\\ \widetilde{A}_n(\varepsilon)&=\left\{\sup_{t_n-T\leq t,s\leq t_n}\left|B^{n+1}(t)-B^{n+1}(s)\right|\leq\varepsilon\right\},\\ \overline{A}_n(\varepsilon)&=\left\{\sup_{t_n-T\leq t,s\leq t_n}\left|\overline{B}^{n+1}(t)-\overline{B}^{n+1}(s)\right|\leq\varepsilon\right\}.\end{aligned}$$

One has $\overline{A}_n(\varepsilon/2)\subset A_n(\varepsilon)\cup(\widetilde{A}_n(\varepsilon/2))^c$ and then

$$\mathbf{1}_{A_n(\varepsilon)}+\mathbf{1}_{(\widetilde{A}_n(\varepsilon/2))^c}\geq\mathbf{1}_{\overline{A}_n(\varepsilon/2)}.$$

We take the conditional expectation with respect to $\mathcal{F}_{t_{n+1}}$ and we deduce that

$$\mathbb{E}\left(\mathbf{1}_{A_n(\varepsilon)}|\mathcal{F}_{t_{n+1}}\right) \geq \mathbb{P}\left(\overline{A}_n(\varepsilon/2)\right) - \mathbf{1}_{(\widetilde{A}_n(\varepsilon/2))^c}$$

because $\overline{A}_n(\varepsilon/2)$ is independent of $\mathcal{F}_{t_{n+1}}$, while $\widetilde{A}_n(\varepsilon/2)$ belongs to $\mathcal{F}_{t_{n+1}}$. Arguing as above we also obtain

$$\mathbb{P}(\overline{A}_n(\varepsilon/2)) + \mathbb{P}((\widetilde{A}_n(\varepsilon/4))^c) \geq \mathbb{P}(A_n(\varepsilon/4)).$$

We add these inequalities and we get

$$\mathbb{E}\left(\mathbf{1}_{A_n(\varepsilon)}|\mathcal{F}_{t_{n+1}}\right) \geq \mathbb{P}(A_n(\varepsilon/4)) - \mathbb{P}((\widetilde{A}_n(\varepsilon/4))^c) - \mathbf{1}_{(\widetilde{A}_n(\varepsilon/2))^c}. \tag{26}$$

We will show hereafter that

$$\sum_{n\geq 1} \mathbb{P}((\widetilde{A}_n(\varepsilon))^c) < \infty, \tag{27}$$

while

$$\mathbb{P}(A_n(\varepsilon)) \geq \exp\left(\frac{-cT}{\varepsilon^H}\right). \tag{28}$$

Assume for a moment that these inequalities hold true. Then from (26) we deduce that $\sum_{n\geq 1} \mathbb{E}(\mathbf{1}_{A_n(\varepsilon)}|\mathcal{F}_{t_{n+1}}) = \infty$ a.s. and by Lemma 9 we obtain $\sum_{n\geq 1} \mathbf{1}_{A_n(\varepsilon)} = \infty$ a.s., which implies the expected result.

Proof of (27). Let $t_n - T \leq s \leq t \leq t_n$. By (21) we have

$$B^{n+1}(t) - B^{n+1}(s) = \mathbb{E}\left[\int_{-\infty}^{t_{n+1}} c_H \left\{(s-r)^{H-\frac{1}{2}} - (t-r)^{H-\frac{1}{2}}\right\} dW_r \middle| \mathcal{F}_{t_{n+1}}\right]$$

and for $p \geq 1$ we obtain

$$\mathbb{E}\left(|B^{n+1}(t) - B^{n+1}(s)|^{2p}\right) \leq c\left(\int_{-\infty}^{t_{n+1}} \left|(s-r)^{H-\frac{1}{2}} - (t-r)^{H-\frac{1}{2}}\right|^2 dr\right)^p.$$

In the above integral we make successively the changes of variables $v = r - s$ and $u = v/(t-s)$. This yields

$$\begin{aligned}\left(\mathbb{E}\left(|B^{n+1}(t) - B^{n+1}(s)|^{2p}\right)\right)^{\frac{1}{p}} &\leq c(t-s)^{2H} \int_{-\infty}^{\frac{t_{n+1}-s}{t-s}} \left|(-u)^{H-\frac{1}{2}} - (1-u)^{H-\frac{1}{2}}\right|^2 du \\ &\leq c(t-s)^{2H} \int_{-\infty}^{\frac{t_{n+1}-s}{t-s}} (-u)^{2H-3} du\end{aligned}$$

where we have used the fact that for $-u$ sufficiently big (and positive), $|(-u)^{H-\frac{1}{2}} - (1-u)^{H-\frac{1}{2}}| \leq c(-u)^{H-\frac{3}{2}}$. The above inequality is then true for sufficiently large n. Finally we obtain that

$$\mathbb{E}\left(|B^{n+1}(t) - B^{n+1}(s)|^{2p}\right) \leq c\left((t-s)(t_n - t_{n+1})^{H-1}\right)^{2p}. \tag{29}$$

Now we use the Garsia–Rodemich–Rumsey inequality (see [8]): let f be a continuous function, and ρ and g two continuous strictly increasing functions on $[0, \infty)$ with $\rho(0) = g(0) = 0$ and

$\lim_{x\to\infty}\rho(x)=\infty$. Then it holds that

$$|f(t)-f(s)|\le 8\int_0^{t-s}\rho^{-1}\left(\frac{4C_{s,t}}{u^2}\right)dg(u)$$
$$\text{with } C_{s,t}=\int_s^t\int_s^t\rho\left(\frac{|f(t')-f(s')|}{g(|t'-s'|)}\right)ds'dt'.$$

We apply the above inequality with $\rho(u)=u^4$ and $g(u)=u$. Thus there exists a constant c and a random variable δ_n such that

$$|B^{n+1}(t)-B^{n+1}(s)|\le\delta_n\times|t-s|^{1/2}\quad\text{with}$$
$$\delta_n=c\left(\int_{t_n-T}^{t_n}\int_{t_n-T}^{t_n}\left(\frac{|B^{n+1}(t')-B^{n+1}(s')|}{|t'-s'|}\right)^4ds'dt'\right)^{1/4}.$$

By (29) and the Jensen inequality, it is clear that

$$\mathbb{E}(|\delta_n|^{2p})\le cT^p(t_n-t_{n+1})^{2p(H-1)},$$

and we obtain

$$\sup_{t_n-T\le t,s\le t_n}\left|B^{n+1}(t)-B^{n+1}(s)\right|\le c\,T^{1/2}\delta_n. \tag{30}$$

Now we write that

$$\begin{aligned}\mathbb{P}((\widetilde{A}_n(\varepsilon))^c)&\le\frac{1}{\epsilon}\mathbb{E}\left(\sup_{t_n-T\le t,s\le t_n}\left|B^{n+1}(t)-B^{n+1}(s)\right|\right)\\&\le c\frac{T^{1/2}}{\epsilon}\mathbb{E}(\delta_n)\le c\frac{T^{1/2}}{\epsilon}\left(\mathbb{E}(\delta_n^2)\right)^{1/2}\\&\le c\frac{T}{\epsilon}(t_n-t_{n+1})^{H-1}\end{aligned}$$

and since $\sum_{n\ge1}(t_n-t_{n+1})^{H-1}<\infty$, we obtain (27).

Proof of (28). This inequality is a consequence of Talagrand's small ball estimate (see [18] or [14, Theorem 3.8]). Indeed, one needs al least $T\varepsilon^{-H}$ balls of radius ε under the Dudley metric $d(s,t)=\left(\mathbb{E}|B(t)-B(s)|^2\right)^{1/2}$ to cover the time interval $[t_n-T,t_n]$. It follows that there exists a constant c such that

$$\log\mathbb{P}\left(\sup_{t_n-T\le t,s\le t_n}|B(t)-B(s)|\le\varepsilon\right)\ge -c\frac{T}{\varepsilon^H}$$

and we deduce (28). This achieves our first step.

Step 2: We prove Theorem 2 for the noise $F(t,x)=\sum_{k\ge1}F_k(x)B_k(t)$. With $c_k=\|F_k\|_{C_b^2(\mathbb{R})}$, we define

$$\mathbf{B}(t)=\sum_{k\ge1}c_kB_k(t).$$

For $t\ge t_{n+1}$, we set $\mathcal{F}_{t_n}=\sigma\{B_k(r);-\infty<r\le t_n;k\ge1\}$ and

$$\mathbf{B}^{n+1}(t)=\mathbb{E}(\mathbf{B}(t)|\mathcal{F}_{t_{n+1}})$$
$$\overline{\mathbf{B}}^{n+1}(t)=\mathbf{B}(t)-\mathbf{B}^{n+1}(t).$$

Replacing B by $\mathbf{B}$ in the events $A_n(\varepsilon)$, $\widetilde{A}_n(\varepsilon)$ and $\overline{A}_n(\varepsilon)$, we define the events $\mathbf{A}_n(\varepsilon)$, $\widetilde{\mathbf{A}}_n(\varepsilon)$ and $\overline{\mathbf{A}}_n(\varepsilon)$ by

$$\mathbf{A}_n(\varepsilon) = \left\{ \sum_{k\geq 1} c_k \sup_{t_n - T \leq t,s \leq t_n} |B_k(t) - B_k(s)| \leq \varepsilon \right\},$$

$$\widetilde{\mathbf{A}}_n(\varepsilon) = \left\{ \sum_{k\geq 1} c_k \sup_{t_n - T \leq t,s \leq t_n} \left| B_k^{n+1}(t) - B_k^{n+1}(s) \right| \leq \varepsilon \right\},$$

$$\overline{\mathbf{A}}_n(\varepsilon) = \left\{ \sum_{k\geq 1} c_k \sup_{t_n - T \leq t,s \leq t_n} \left| \overline{B}_k^{n+1}(t) - \overline{B}_k^{n+1}(s) \right| \leq \varepsilon \right\}.$$

Clearly (6) will be proved as soon as the inequalities (27) and (28) are replaced by

$$\sum_{n\geq 1} \mathbb{P}((\widetilde{\mathbf{A}}_n(\varepsilon))^c) < \infty \quad \text{and} \tag{31}$$

$$\mathbb{P}(\mathbf{A}_n(\varepsilon)) \geq \exp\left(\frac{-cT}{\varepsilon^H}\right). \tag{32}$$

The inequality (30) is valid for any of the fractional Brownian motion B_k. Thus for any $k \geq 1$

$$\sup_{t_n - T \leq t,s \leq t_n} \left| B^{n+1}(t) - B^{n+1}(s) \right| \leq c\, T^{1/2} \delta_n$$

and we deduce that

$$\begin{aligned} \mathbb{P}\big((\widetilde{\mathbf{A}}_n(\varepsilon))^c\big) &\leq \frac{1}{\varepsilon} \mathbb{E}\left(\sum_{k\geq 1} c_k \sup_{t_n - T \leq t,s \leq t_n} \left| B_k^{n+1}(t) - B_k^{n+1}(s) \right| \right) \\ &\leq c\, T^{1/2} \left(\sum_{k\geq 1} c_k \right) \mathbb{E}(\delta_n) \\ &\leq c\, T (t_n - t_{n+1})^{H-1} \left(\sum_{k\geq 1} c_k \right). \end{aligned}$$

We use $\sum_{k\geq 1} c_k < \infty$ (Hypothesis III(a)) and (31) holds true.

Now we prove (32). We repeat the arguments of the proof of (28). We have for any $k, n \geq 1$

$$\mathbb{P}\left(\sup_{t_n - T \leq t,s \leq t_n} |B_k(t) - B_k(s)| \leq \frac{\varepsilon k^{2/H}}{\mathfrak{C}} \right) \geq \exp\left\{ -c \frac{T \mathfrak{C}^H}{\varepsilon^H k^2} \right\},$$

with $\mathfrak{C} = \sum_{k\geq 1} c_k k^{2/H} < \infty$ (Hypothesis III(a)). For each n the events

$$\mathfrak{A}_{n,k}(\varepsilon) = \left\{ \sup_{t_n - T \leq t,s \leq t_n} |B_k(t) - B_k(s)| \leq \frac{\varepsilon k^{2/H}}{\mathfrak{C}} \right\}, \quad k \geq 1$$

are independent and $\cap_{k\geq 1} \mathfrak{A}_{n,k}(\varepsilon) \subseteq \mathbf{A}_n(\varepsilon)$. Then

$$\mathbb{P}(\mathbf{A}_n(\varepsilon)) \geq \prod_{k\geq 1} \mathbb{P}\left(\mathfrak{A}_{n,k}(\varepsilon)\right) \geq \exp\left\{ -c\, \frac{T \mathfrak{C}^H}{\varepsilon^H} \sum_{k\geq 1} \frac{1}{k^2} \right\} > 0$$

and (32) is proved. This completes our proof. □

Appendix A. Proof of Proposition 4

Step 1: *W satisfies the Hamilton–Jacobi–Bellman equation* (13). We define

$$B_R(t_1,t_2)=\left\{\xi\in H^1(t_1,t_2);\,|\xi(t_1)|+\int_{t_1}^{t_2}|\dot\xi(s)|^2ds\le R\right\}$$

which is clearly a closed and bounded subset of $H^1(t_1,t_2)$, and hence weakly compact. Now we prove that there exists on $B_R(t_0,t)$ one minimizer of $\xi\mapsto F(\xi):=\widetilde{A}_{t_0,t}(\xi)+U_0(\xi(t_0))$. By the weak compactness of $B_R(t_0,t)$ it is sufficient that $\xi\mapsto F(\xi)$ is lower semi-continuous. Following [7, Theorem I.9.1] we just have to check the lower semi-continuity of the stochastic part

$$S(\xi)=-\sum_{k\ge1}\int_{t_0}^{t}(B_k(s)-B_k(t_0))F_k'(\xi(s))\dot\xi(s)ds.$$

Let $(\xi_n)_{n\ge1}$ be a sequence of $B_R(t_0,t)$ converging to ξ weakly. The weak convergence on $B_R(t_0,t)$ implies the uniform convergence on $[t_0,t]$. Writing $S(\xi)-S(\xi_n)=S_n^1+S_n^2$ with

$$S_n^1=\sum_{k\ge1}\int_{t_0}^{t}(B_k(s)-B_k(t_0))\big[F_k'(\xi_n(s))-F_k'(\xi(s))\big]\dot\xi_n(s)ds$$
$$S_n^2=\sum_{k\ge1}\int_{t_0}^{t}(B_k(s)-B_k(t_0))F_k'(\xi(s))\big[\dot\xi_n(s)-\dot\xi(s)\big]ds,$$

and by uniform convergence, $\lim_n S_n^1=0$. The weak convergence and the fact that $s\mapsto\sum_{k\ge1}(B_k(s)-B_k(t_0))F_k'(\xi(s))$ belongs to $L^2(t_0,t)$ yield $\lim_n S_n^2=0$.

Hence we have the lower semi-continuity and thus there exists a minimizer $\xi_{\min}\in B_R(t_0,t)$ of $\xi\mapsto\widetilde{A}_{t_0,t}(\xi)+U_0(\xi(t_0))$. So for every t,x, there exists a minimizer $\xi_{\min}\in H^1(t_0,t)$ with $\xi_{\min}(t)=x$ such that

$$\begin{aligned}W(t,x)&=\inf_{\substack{\xi\in H^1(t_0,t)\\ \xi(t)=x}}\left\{\widetilde{\mathcal{A}}_{t_0,t}(\xi)+U_0(\xi(t_0))\right\}\\&=\int_{t_0}^{t}L(s,\xi_{\min}(s),\dot\xi_{\min}(s))ds+U_0(\xi_{\min}(t_0)).\end{aligned}\tag{33}$$

Working with the right end-point condition $\xi(t)=x$ in the calculus of variations will not affect Theorems I.9.2, I.9.3 and I.9.4 of [7]. Thus there exists M such that for any (t,x) and (t',x') in $\mathbb{R}\times\mathbb{R}$,

$$|W(t,x)-W(t',x')|\le M\big(|t-t'|+|x-x'|\big).\tag{34}$$

The equation satisfied by W will be obtained thanks to the following version of the dynamic programming principle. Indeed we can observe that for any $t_0\le r\le t$,

$$W(t,x)=\inf_{\substack{\xi\in H^1(t_0,t)\\ \xi(t)=x}}\left(\int_r^t L(s,\xi(s),\dot\xi(s))ds+W(r,\xi(r))\right).$$

Now let $0 < h < t - t_0$ and take $r = t - h$ in the above identity. We subtract $W(t,x)$ from both sides and we get

$$\inf_{\substack{\xi \in H^1(t_0,t)\\ \xi(t)=x}} \left(\frac{1}{h} \int_{t-h}^{t} L(s, \xi(s), \dot{\xi}(s)) ds + \frac{1}{h}\big(W(t-h, \xi(t-h)) - W(t,x)\big) \right) = 0.$$

When $h \downarrow 0$, we obtain

$$\begin{aligned}
&-\frac{\partial W}{\partial t}(t,x) + \inf_{\substack{\xi \in H^1(t_0,t)\\ \xi(t)=x}} \left(L(t,x,\dot{\xi}(t)) - \frac{\partial W}{\partial x}(t,x) \times \dot{\xi}(t) \right) = 0\\
&+\frac{\partial W}{\partial t}(t,x) - \inf_{q\in\mathbb{R}} \left(-q \times \frac{\partial W}{\partial x}(t,x) + L(t,x,q) \right) = 0\\
&+\frac{\partial W}{\partial t}(t,x) + \sup_{q\in\mathbb{R}} \left(+q \times \frac{\partial W}{\partial x}(t,x) - L(t,x,q) \right) = 0\\
&+\frac{\partial W}{\partial t}(t,x) + H\left(t, x, \frac{\partial W}{\partial x}(t,x) \right) = 0
\end{aligned}$$

where $p \mapsto H(t,x,p)$ is the Legendre transform of $q \mapsto L(t,x,q)$. Using the behavior under translation of the Legendre transform, we have $H(t,x,p) = \Psi(p + v(t,x))$ where v is defined in (11). In other words, for all t, x we have that W satisfies the Hamilton–Jacobi–Bellman equation (13) (also referred to in the literature as the dynamic programming equation).

Step 2: *semi-concavity*.

The concavity of $x \mapsto W(t,x) - Kx^2$ is equivalent to

$$W(t,x) \geq \frac{1}{2}\big(W(t,x+h) + W(t,x-h)\big) - K\left(1 + \frac{1}{t - t_0}\right) \times h^2, \quad \forall x, h. \tag{35}$$

Let $\xi_{\min}$ be the minimizer of the action such that W satisfies (33) (we recall that $\xi_{\min}(t) = x$). We introduce γ_{x+h} and γ_{x-h} in $H^1(t_0,t)$ defined by

$$\gamma_{x\pm h}(s) = \xi_{\min}(s) \pm \frac{s - t_0}{t - t_0} h,$$

thus satisfying $\gamma_{x\pm h}(t) = x \pm h$ and $\gamma_{x\pm h}(t_0) = \xi_{\min}(t_0)$. We calculate

$$\begin{aligned}
\Delta^1_{x,h} &= W(t,x+h) + W(t,x-h)\\
&\leq \int_{t_0}^{t} \big(L(s, \gamma_{x+h}(s), \dot{\gamma}_{x+h}(s)) + L(s, \gamma_{x-h}(s), \dot{\gamma}_{x-h}(s)) \big) ds\\
&\quad + U_0(\gamma_{x+h}(t_0)) + U_0(\gamma_{x-h}(t_0))\\
&\leq \int_{t_0}^{t} \big(L(s, \xi_{\min}(s), \dot{\gamma}_{x+h}(s)) + L(s, \xi_{\min}(s), \dot{\gamma}_{x-h}(s)) \big) ds\\
&\quad + \int_{t_0}^{t} \big(L(s, \gamma_{x+h}(s), \dot{\gamma}_{x+h}(s)) - L(s, \xi_{\min}(s), \dot{\gamma}_{x+h}(s)) \big) ds\\
&\quad + \int_{t_0}^{t} \big(L(s, \gamma_{x-h}(s), \dot{\gamma}_{x-h}(s)) - L(s, \xi_{\min}(s), \dot{\gamma}_{x-h}(s)) \big) ds + 2U_0(\xi_{\min}(t_0))\\
&\leq \delta^1_{x,h} + \delta^2_{x,h} + \delta^3_{x,h} + 2U_0(\xi_{\min}(t_0)),
\end{aligned}$$

with obvious notation. First we evaluate the term $\delta^1_{x,h}$. We recall that since Ψ is uniformly convex, for any real q we have $\Psi''(q) \geq \theta$. Then the Legendre transform $L(s,x,p) = \Psi(\cdot, \gamma_{x+h}(s))^*(p)$ satisfies (see [5, page 131])

$$\frac{1}{2}L(s,x,p_1) + \frac{1}{2}L(s,x,p_2) \leq L\big(s,x,(p_1+p_2)/2\big) + \frac{1}{8\theta}|p_1 - p_2|^2.$$

Using the identities $\dot{\gamma}_{x+h} + \dot{\gamma}_{x-h} = 2\dot{\xi}_{\min}$ and $\dot{\gamma}_{x+h} - \dot{\gamma}_{x-h} = 2h/(t-t_0)$, we deduce that

$$\begin{aligned}\delta^1_{x,h} &\leq 2\int_{t_0}^t \left\{L\big(s, \xi_{\min}(s), (\dot{\gamma}_{x+h}(s) + \dot{\gamma}_{x-h}(s))/2\big) + C|\dot{\gamma}_{x+h}(s) - \dot{\gamma}_{x-h}(s)|^2\right\} ds \\ &\leq 2\int_{t_0}^t L(s, \xi_{\min}(s), \dot{\xi}_{\min}(s))ds + C\frac{h^2}{t-t_0}.\end{aligned}$$

We finally obtain that

$$\delta^1_{x,h} + 2U_0(\xi_{\min}(t_0)) \leq 2W(t,x) + C\frac{h^2}{t-t_0}.$$

Now we write

$$\begin{aligned}\delta^2_{x,h} &= \int_{t_0}^t \sum_{k\geq 1}(B_k(s) - B_k(t_0))\big[F_k'(\gamma_{x+h}(s)) - F_k'(\xi_{\min}(s))\big]\dot{\gamma}_{x+h}(s)ds \\ &= \int_{t_0}^t \Bigg\{\sum_{k\geq 1}(B_k(s) - B_k(t_0))\dot{\gamma}_{x+h}(s) \\ &\quad \times \left(\int_0^1 F_k''\left((1-\nu)\gamma_{x+h}(s) - \nu\xi_{\min}(s)\right)\left(\gamma_{x+h}(s) - \xi_{\min}(s)\right)d\nu\right)\Bigg\}ds \\ &= \int_{t_0}^t \Bigg\{\sum_{k\geq 1}(B_k(s) - B_k(t_0))\left(\int_0^1 F_k''\left(\xi_{\min}(s) + (1-\nu)\frac{s-t_0}{t-t_0}h\right)d\nu\right) \\ &\quad \times \left(\dot{\xi}_{\min}(s) + \frac{h}{t-t_0}\right)\frac{s-t_0}{t-t_0}h\Bigg\}ds\end{aligned}$$

and analogously it holds that

$$\begin{aligned}\delta^3_{x,h} &= \int_{t_0}^t \Bigg\{\sum_{k\geq 1}(B_k(s) - B_k(t_0))\left(\int_0^1 F_k''\left(\xi_{\min}(s) - (1-\nu)\frac{s-t_0}{t-t_0}h\right)d\nu\right) \\ &\quad \times \left(\dot{\xi}_{\min}(s) - \frac{h}{t-t_0}\right)\frac{s-t_0}{t-t_0}(-h)\Bigg\}ds.\end{aligned}$$

We compute the sum

$$\begin{aligned}
\delta^2_{x,h}+\delta^3_{x,h} &= \int_{t_0}^{t}\Bigg\{\sum_{k\geq 1}(B_k(s)-B_k(t_0))\frac{h^2(s-t_0)}{t-t_0}\\
&\quad\times\left(\int_0^1\left[F_k''\left(\xi_{\min}(s)+(1-\nu)\frac{s-t_0}{t-t_0}h\right)\right.\right.\\
&\quad\left.\left.+\;F_k''\left(\xi_{\min}(s)-(1-\nu)\frac{s-t_0}{t-t_0}h\right)\right]d\nu\right)\Bigg\}ds\\
&\quad+\int_{t_0}^{t}\Bigg\{\sum_{k\geq 1}(B_k(s)-B_k(t_0))\frac{h(s-t_0)}{t-t_0}\dot{\xi}_{\min}(s)\\
&\quad\times\left(\int_0^1\left[F_k''\left(\xi_{\min}(s)+(1-\nu)\frac{s-t_0}{t-t_0}h\right)\right.\right.\\
&\quad\left.\left.-\;F_k''\left(\xi_{\min}(s)-(1-\nu)\frac{s-t_0}{t-t_0}h\right)\right]d\nu\right)\Bigg\}ds
\end{aligned}$$

and using Hypothesis I and the identity

$$\begin{aligned}
&\int_0^1\left[F_k''\left(\xi_{\min}(s)+(1-\nu)\frac{s-t_0}{t-t_0}h\right)-F_k''\left(\xi_{\min}(s)-(1-\nu)\frac{s-t_0}{t-t_0}h\right)\right]d\nu\\
&\quad=\int_0^1\int_0^1 F_k'''\left(\xi_{\min}(s)+(1-2\mu)(1-\nu)\frac{s-t_0}{t-t_0}h\right)d\mu\; 2(1-\nu)\frac{s-t_0}{t-t_0}h\,d\mu
\end{aligned}$$

we deduce that

$$\begin{aligned}
\delta^2_{x,h}+\delta^3_{x,h} &\leq 2(t-t_0)^{\lambda+1}\sum_{k\geq 1}\|F_k''\|_\infty\|B_k\|_{t_0,t,\lambda}\times h^2\\
&\quad+(t-t_0)^{\lambda}\sum_{k\geq 1}\|F_k'''\|_\infty\|B_k\|_{t_0,t,\lambda}\times h^2\times\|\xi_{\min}\|_{H^1(t_0,t)}\\
&\leq C\times h^2.
\end{aligned}$$

As a conclusion we obtain (35).

Remark. By Alexandrov's theorem (see Appendix E in [7]), $x\mapsto W(t,x)$ is almost everywhere twice differentiable.

Appendix B. Proof of Proposition 5

Proof of (a). We define

$$K^{B,F}_{t_1,t_2}=\sum_{k\geq 1}\|F_k\|_{C^3}\left\{\sup_{t_1\leq r\leq s\leq t_2}|B_k(s)-B_k(r)|\right\}.$$

For $t_1 < t < t_2$ the operator $\mathcal{L} : C^1(t, t_2) \to C^1(t, t_2)$ is defined by

$$\begin{cases} \dot{\mathcal{L}(\xi)} = \Psi'(v) \\ v(s) = v(t_2) - \displaystyle\int_s^{t_2} \sum_{k\geq 1} \big(B_k(r) - B_k(s)\big) F_k''(\xi(r))\dot{\xi}(r)dr \\ \qquad + \displaystyle\sum_{k\geq 1} \big(B_k(t_2) - B_k(s)\big) F_k'(\xi(t_2)) \end{cases}$$

with $\mathcal{L}(\xi)(t_2) = \xi(t_2)$ and $\dot{\mathcal{L}(\xi)}(t_2) = \Psi'(v_2) = \dot{\xi}(t_2)$. We have

$$\begin{aligned} \|\dot{\mathcal{L}(\xi)}\|_{t,t_2,\infty} &\leq |\Psi'(v(t_2))| + \|\Psi'(v) - \Psi'(v(t_2))\|_{t,t_2,\infty} \\ &\leq |\dot{\xi}(t_2)| + L\|v - v(t_2)\|_{t,t_2,\infty} \\ &\leq |\dot{\xi}(t_2)| + L(t_2 - t_1)^{\lambda} K_{t_1,t_2}^{B,F} \big((t_2 - t_1)\|\dot{\xi}\|_{t,t_2,\infty} + 1\big) \end{aligned}$$

and since $\mathcal{L}(\xi)(s) = \xi(t_2) + \int_s^{t_2} \dot{\mathcal{L}(\xi)}(r)dr$ we may write

$$\|\mathcal{L}(\xi)\|_{t,t_2,\infty} \leq |\xi(t_2)| + C_{t_1,t_2}(t_2 - t)\big(1 + \|\dot{\xi}\|_{t,t_2,\infty}\big).$$

Consequently $\|\mathcal{L}(\xi)\|_{C^1(t,t_2)} \leq |\xi_2| + |\Psi'(v_2)| + C(t_2 - t)(1 + \|\dot{\xi}\|_{C^1(t,t_2)})$ and the operator $\mathcal{L}$ satisfies $\mathcal{L}(\mathcal{B}_0) \subseteq \mathcal{B}_0$ with

$$\mathcal{B}_0 = \{\xi \in C^1(t, t_2) : \|\xi\|_{C^1(t,t_2)} \leq 2(1 + |\xi_2| + |\Psi'(v_2)|)\}$$

provided that t is small enough to ensure that $C(t_2 - t) \leq 1/2$. Let $\gamma_1, \gamma_2 \in \mathcal{B}_0$ and $\nu_i = (\Psi')^{-1}(\dot{\gamma}_i)$ for $i = 1, 2$. The following identity:

$$\begin{aligned} \nu_1(s) - \nu_2(s) = &- \int_s^{t_2} \sum_{k\geq 1} \big(B_k(r) - B_k(s)\big) F_k''(\gamma_1(r))\big[\dot{\gamma}_1(r) - \dot{\gamma}_2(r)\big]dr \\ &- \int_s^{t_2} \sum_{k\geq 1} \big(B_k(r) - B_k(s)\big)\big[F_k''(\gamma_1(r)) - F_k''(\gamma_2(r))\big]\dot{\gamma}_2(r) \end{aligned}$$

implies that

$$\|\mathcal{L}(\gamma_1) - \mathcal{L}(\gamma_2)\|_{C^1(t,t_2)} \leq C(t_2 - t)\|\gamma_1 - \gamma_2\|_{C^1(t,t_2)}.$$

Hence $\mathcal{L}$ is a contraction on $\mathcal{B}_0$ (with t eventually smaller) and there exists $\xi \in \mathcal{B}_0$ such that $\mathcal{L}(\xi) = \xi$. Then there exists a unique solution in $C^1(t, t_2)$ to the Euler–Lagrange equations (15) for short time. By a concatenation argument, the existence and uniqueness is extended to $C^1(t_1, t_2)$. $\square$

Proof of (b). Since γ minimizes the functional $\mathcal{A}_{t_1,t_2}$, we have that for any $\xi \in \mathcal{H}_{x_1,x_2}^{t_1,t_2}$, $\varepsilon \mapsto \frac{d}{d\varepsilon}\mathcal{A}_{t_1,t_2}(\gamma + \varepsilon\xi)$ equals 0 in $\varepsilon = 0$. This yields

$$\begin{aligned} 0 = &\int_{t_1}^{t_2} \Big[(\Psi^*)'(\dot{\gamma})(s)\dot{\xi}(s) - \sum_{k\geq 1} (B_k(s) - B_k(t_1))\big(F_k''(\gamma)\dot{\gamma}\xi + F_k'(\gamma)\dot{\xi}\big)(s) \Big] ds \\ &+ \sum_{k\geq 1} (B_k(t_2) - B_k(t_1)) F_k'(\gamma(t_2))\xi(t_2). \end{aligned}$$

For $t_1 < \tau_1 \le \tau_2 < t_2$, we write this identity with ξ_n defined as

$$\xi_n(s) = 0 \times \mathbf{1}_{[t_1,\tau_1]\cup[\tau_2,t_2]}(s) + n\big(s - (\tau_1 - 1/n)\big)\mathbf{1}_{[\tau_1-1/n,\tau_1]}(s) + \mathbf{1}_{[\tau_1,\tau_2]}(s) + n\big(-s + (\tau_2 + 1/n)\big)\mathbf{1}_{[\tau_2,\tau_2+1/n]}(s).$$

We obtain

$$\begin{aligned}
&\int_{\tau_2}^{\tau_2+1/n} n(\Psi^*)'(\dot\gamma(s))ds - \int_{\tau_1-1/n}^{\tau_1} n(\Psi^*)'(\dot\gamma(s))ds \\
&= -\int_{\tau_1}^{\tau_2} \sum_{k\ge1}(B_k(s) - B_k(t_1))F_k''(\gamma(s))\dot\gamma(s)ds \\
&\quad - \int_{\tau_1-1/n}^{\tau_1} \sum_{k\ge1}(B_k(s) - B_k(t_1))(F_k''(\gamma)\dot\gamma\xi_n)(s)ds \\
&\quad - \int_{\tau_1-1/n}^{\tau_1} n\sum_{k\ge1}(B_k(s) - B_k(t_1))F_k'(\gamma(s))ds \\
&\quad - \int_{\tau_2}^{\tau_2+1/n} \sum_{k\ge1}(B_k(s) - B_k(t_1))(F_k''(\gamma)\dot\gamma\xi_n)(s)ds \\
&\quad + \int_{\tau_2}^{\tau_2+1/n} n\sum_{k\ge1}(B_k(s) - B_k(t_1))F_k'(\gamma(s))ds.
\end{aligned}$$

We remark that $\sup_n \|\xi_n\|_\infty \le c$ and easy arguments allow us to let n go to infinity. Hence

$$\begin{aligned}
(\Psi^*)'(\dot\gamma(\tau_2)) - (\Psi^*)'(\dot\gamma(\tau_1)) &= -\int_{\tau_1}^{\tau_2} \sum_{k\ge1}(B_k(s) - B_k(t_1))F_k''(\gamma(s))\dot\gamma(s)ds \\
&\quad + \sum_{k\ge1}(B_k(\tau_2) - B_k(t_1))F_k'(\gamma(\tau_2)) \\
&\quad - \sum_{k\ge1}(B_k(\tau_1) - B_k(t_1))F_k'(\gamma(\tau_1))
\end{aligned}$$

which implies that $\tau \mapsto (\Psi^*)'(\dot\gamma(\tau))$ is continuous and since $(\Psi^*)' = (\Psi')^{-1}$, $\tau \mapsto \dot\gamma(\tau)$ is also continuous. Consequently, and with

$$g(s) = \sum_{k\ge1}(B_k(s) - B_k(t_1))F_k''(\gamma(s))\dot\gamma(s)$$

and the integration by parts formula (9), one may write

$$
\begin{aligned}
(\Psi^*)'(\dot\gamma(\tau_2)) - (\Psi^*)'(\dot\gamma(\tau_1)) &= -\int_{t_1}^{\tau_2} g(s)ds + g(\tau_2) - g(t_1) \\
&\quad - \left(-\int_{t_1}^{\tau_1} g(s)ds + g(\tau_1) - g(t_1)\right) \\
&= \int_{t_1}^{\tau_2} \sum_{k\geq 1} F_k'(\gamma(s))dB_k(s) - \int_{t_1}^{\tau_1} \sum_{k\geq 1} F_k'(\gamma(s))dB_k(s) \\
&= \int_{\tau_1}^{\tau_2} \sum_{k\geq 1} F_k'(\gamma(s))dB_k(s).
\end{aligned}
$$

By the continuity of $\tau \mapsto \int_{t_1}^{\tau} \sum_{k\geq 1} F_k'(\gamma(s))dB_k(s)$ (see Prop. 4.4.1 in [22]), the above formula is also true for $\tau_1 = t_1$ and $\tau_2 = t_2$. Thus the formula (16) is true and $\gamma \in C^1(t_1, t_2)$. □

Proof of (c). We recall that Ψ' is Lipschitz (Hypothesis II(a)) and the Legendre transform of Ψ satisfies also the linear growth condition $c_1|v|^{1+\alpha} \leq |\Psi^*(v)| \leq c_2|v|^{1+\beta}$ with $\alpha = 1/k_2, \beta = 1/k_1$ and two positive constants c_3 and c_4 different from those in Hypothesis II(b)).

Let $t_1 \leq t \leq t_2$ and s be such that $|\dot\gamma(s)| = \inf_{r\in[t_1,t_2]} |\dot\gamma(r)|$. Writing $\dot\gamma(t) = (\Psi' \circ (\Psi')^{-1})(\dot\gamma(t)) - (\Psi' \circ (\Psi')^{-1})(\dot\gamma(s)) + \dot\gamma(s)$, we have

$$
\begin{aligned}
|\dot\gamma(t)| &\leq L|(\Psi')^{-1}(\dot\gamma(t)) - (\Psi')^{-1}(\dot\gamma(s))| + |\dot\gamma(s)| \\
&\leq L \times \Delta_{s,t} + \frac{\|\dot\gamma\|_{L^1(t_1,t_2)}}{t_2 - t_1},
\end{aligned}
$$

with

$$
\begin{aligned}
\Delta_{s,t} &= |(\Psi')^{-1}(\dot\gamma(t)) - (\Psi')^{-1}(\dot\gamma(s))| \\
&= -\int_s^t \sum_{k\geq 1}\big(B_k(r) - B_k(s)\big)F_k''(\gamma(r))\dot\gamma(r)dr + \sum_{k\geq 1}\big(B_k(t) - B_k(s)\big)F_k'(\gamma(t)) \\
&\leq C_{t_1,t_2} + C_{t_1,t_2}\|\dot\gamma\|_{L^1(t_1,t_2)}.
\end{aligned}
$$

Consequently,

$$
|\dot\gamma(t)| \leq C_{t_1,t_2}L + \big(C_{t_1,t_2}L + 1/(t_2 - t_1)\big)\,\|\dot\gamma\|_{L^1(t_1,t_2)}. \tag{36}
$$

Now we estimate the L^1 norm of $\dot\gamma$. We recall that $c_1|v|^{1+\alpha} \leq |\Psi^*(v)|$. By Young's inequality $ab \leq (c_1/2)\, a^{1+\alpha} + cb^{(1+\alpha)/\alpha}$ and Jensen's inequality we obtain

$$
\int_{t_1}^{t_2} \sum_{k\geq 1}\big(B_k(r) - B_k(s)\big)F_k''(\gamma(r))\dot\gamma(r)dr \leq c\,(t_2 - t_1)C_{t_1,t_2}^{(1+\alpha)/\alpha} + \frac{c_1}{2}\int_{t_1}^{t_2} |\dot\gamma(s)|^{1+\alpha}ds.
$$

Since γ is a minimizer,

$$\mathcal{A}_{t_1,t_2}(\gamma) = \int_{t_1}^{t_2} \Psi^*(\dot{\gamma}(s)) - \sum_{k\geq 1}(B_k(s) - B_k(t_1))F_k'(\gamma(s))\dot{\gamma}(s)ds + \sum_{k\geq 1}(B_k(t_2) - B_k(t_1))F_k(\gamma(t_2))$$

and

$$\frac{c_1}{2}\int_{t_1}^{t_2} |\dot{\gamma}(s)|^{1+\alpha}ds \leq \mathcal{A}_{t_1,t_2}(\gamma) + c\,(t_2 - t_1)C_{t_1,t_2}^{(1+\alpha)/\alpha} + C_{t_1,t_2}. \tag{37}$$

By the minimization property of γ, $\mathcal{A}_{t_1,t_2}(\gamma) \leq \mathcal{A}_{t_1,t_2}(\xi)$ with the curve ξ defined by $\xi(s) = x_1 + (s - t_1)/(t_2 - t_1) \times (x_2 - x_1)$. Using $|\Psi^*(v)| \leq c_2|v|^{1+\beta}$ we may write

$$\mathcal{A}_{t_1,t_2}(\gamma) \leq c\,C_{t_1,t_2} + c\frac{(x_2 - x_1)^{1+\beta}}{(t_2 - t_1)^{\beta}} \leq c\,((x_2 - x_1)^{1+\beta} + C_{t_1,t_2})$$

where we used the fact that $t_2 - t_1 \geq 1$. We report the above inequality in (37) and we get that

$$\int_{t_1}^{t_2} |\dot{\gamma}(s)|^{1+\alpha}ds \leq c\left((x_2 - x_1)^{1+\beta} + C_{t_1,t_2} + C_{t_1,t_2}^{(1+\alpha)/\alpha}(t_2 - t_1)\right).$$

Since $\|\dot{\gamma}\|_{L^1(t_1,t_2)} \leq (t_2 - t_1)^{\alpha/(1+\alpha)}\|\dot{\gamma}\|_{L^{1+\alpha}(t_1,t_2)}$, with (36) we obtain

$$|\dot{\gamma}(t)| \leq c\,C_{t_1,t_2} + c\left(C_{t_1,t_2} + \frac{1}{t_2 - t_1}\right)(t_2 - t_1)^{\alpha/(1+\alpha)} \times \left((x_2 - x_1)^{1+\beta} + C_{t_1,t_2} + C_{t_1,t_2}^{(1+\alpha)/\alpha}(t_2 - t_1)\right)^{1/(1+\alpha)}$$

and using the inequality $(1 + x)^a \leq 1 + x^a$ when $a < 1$ and $x \geq 0$, we deduce (17). □

References

[1] C.M. Dafermos, Generalized characteristics and the structure of solutions of hyperbolic conservation laws, Indiana Univ. Math. J. 26 (6) (1977) 1097–1119.

[2] C.M. Dafermos, Hyperbolic Conservation Laws in Continuum Physics, second ed., in: Grundlehren der Mathematischen Wissenschaften (Fundamental Principles of Mathematical Sciences), vol. 325, Springer-Verlag, Berlin, 2005.

[3] A. Debussche, J. Vovelle, Scalar conservation laws with stochastic forcing, J. Funct. Anal. 259 (4) (2010) 1014–1042.

[4] J.L. Doob, Stochastic Processes, John Wiley & Sons Inc., New York, 1953.

[5] L.C. Evans, Partial Differential Equations, in: Graduate Studies in Mathematics, vol. 19, American Mathematical Society, Providence, RI, 1998.

[6] J. Feng, D. Nualart, Stochastic scalar conservation laws, J. Funct. Anal. 255 (2) (2008) 313–373.

[7] W.H. Fleming, H.M. Soner, Controlled Markov Processes and Viscosity Solutions, in: Applications of Mathematics (New York), vol. 25, Springer-Verlag, New York, 1993.

[8] A.M. Garsia, E. Rodemich, H. Rumsey Jr., A real variable lemma and the continuity of paths of some Gaussian processes, Indiana Univ. Math. J. 20 (1970–1971) 565–578.

[9] H. Holden, N.H. Risebro, Conservation laws with a random source, Appl. Math. Optim. 36 (2) (1997) 229–241.

[10] L. Hörmander, Lectures on Nonlinear Hyperbolic Differential Equations, in: Mathématiques & Applications (Berlin) (Mathematics & Applications), vol. 26, Springer-Verlag, Berlin, 1997.

[11] H. Hult, Approximating some Volterra type stochastic integrals with applications to parameter estimation, Stochastic Process. Appl. 105 (1) (2003) 1–32.

[12] J.U. Kim, On a stochastic scalar conservation law, Indiana Univ. Math. J. 52 (1) (2003) 227–256.
[13] P.D. Lax, Hyperbolic systems of conservation laws. II, Comm. Pure Appl. Math. 10 (1957) 537–566.
[14] W.V. Li, Q.-M. Shao, Gaussian processes: inequalities, small ball probabilities and applications, in: Stochastic Processes: Theory and Methods, in: Handbook of Statist., vol. 19, North-Holland, Amsterdam, 2001, pp. 533–597.
[15] O.A. Oleĭnik, Discontinuous solutions of non-linear differential equations, Uspekhi Mat. Nauk (NS) 12 (3(75)) (1957) 3–73.
[16] D. Serre, Systems of Conservation Laws. 1: Hyperbolicity, Entropies, Shock Waves, Cambridge University Press, Cambridge, 1999, Translated from the 1996 French original by I.N. Sneddon.
[17] W.F. Stout, Almost Sure Convergence, in: Probab. Math. Statist., vol. 24, Academic Press, New York, London, 1974, [A Subsidiary of Harcourt Brace Jovanovich, Publishers].
[18] M. Talagrand, Transportation cost for Gaussian and other product measures, Geom. Funct. Anal. 6 (3) (1996) 587–600.
[19] G. Vallet, P. Wittbold, On a stochastic first order hyperbolic equation in a bounded domain, Infin. Dimens. Anal. Quantum Probab. Relat. Top. 12 (4) (2009) 613–651.
[20] Weinan E, K. Khanin, A. Mazel, Y. Sinai, Invariant measures for Burgers equation with stochastic forcing, Ann. of Math. (2) 151 (3) (2000) 877–960.
[21] L.C. Young, An inequality of the Hölder type connected with Stieltjes integration, Acta Math. 67 (1936) 251–282.
[22] M. Zähle, Integration with respect to fractal functions and stochastic calculus. I, Probab. Theory Related Fields 111 (3) (1998) 333–374.