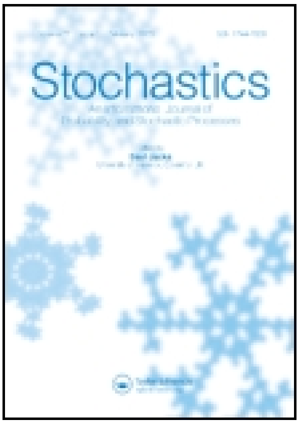

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A remark on the mean square distance between the solutions of fractional SDEs and Brownian SDEs

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We study the family of solutions of differential equations driven by fractional Brownian motions when the Hurst parameter varies between 1/2 and 1. The drift and the diffusion coefficient may also vary in a family of differentiable functions. We prove that there exists a finite covering of this set of solutions by open balls of $\mathbb{L}^2([0,T]\times\Omega)$ centred in some solutions of classical stochastic differential equations driven by a Brownian motion.

Keywords: stochastic differential equations; fractional Brownian motion; Malliavin calculus; fractional calculus

AMS Subject Classification: 60H05; 60H07

1. Introduction

Let H be a parameter that varies in (H_0, H_1) with $1/2 < H_0 \leq H_1 < 1$. Suppose that $B^H = (B_t^H)_{t\in[0,T]}$ is an m-dimensional fractional Brownian motion (fBm in short) with Hurst parameter H. We mean that the components $B^{H,j}$, $j = 1, \ldots, m$, are independent centred Gaussian processes with the covariance function

$$R_H(s,t) = \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right). \tag{1}$$

It is well known that B^H has α-Hölder continuous paths for all $\alpha \in (0,H)$ (we refer to [13] and references therein for further information about fBm and stochastic integration with respect to this process).

We temporarily fix $1/2 < H < 1$ and we consider the solution $\{X_t, t \in [0,T]\}$ of the following stochastic differential equation (SDE) on $\mathbb{R}^d$:

$$X_t^i = x_0^i + \sum_{j=1}^m \int_0^t \sigma^{i,j}(X_s)\mathrm{d}B_s^{H,j} + \int_0^t b^i(X_s)\mathrm{d}s, \quad t \in [0,T], \tag{2}$$

$i = 1, \ldots, d$, $x_0 \in \mathbb{R}^d$ is the initial value of the process X.

Under suitable assumptions on σ, the processes $(\sigma(X_s))_{s\in[0,T]}$ and B^H have trajectories that are Hölder continuous of order strictly larger than $1/2$, so we can use the integral introduced by Young [17]. The stochastic integral in (2) is then a path-wise Riemann–Stieltjes integral. A first result on the existence and uniqueness of a solution of such an

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equation was obtained in [11] using the notion of p-variation. The theory of rough paths introduced by Lyons [11] was used by Coutin and Qian in order to prove an existence and uniqueness result for equation (2) (see [3]). The Riemann–Stieltjes integral appearing in equation (2) can be expressed as a Lebesgue integral using a fractional integration by parts formula (see Zähle [18]). Using this formula, Nualart and Răşcanu have established in [14] the existence of a unique solution for a class of general differential equations that includes (2). The Malliavin regularity and its application to the absolute continuity of the law of X_t have been studied in [2,9,12,15]. The flow property of the solution of (2) is studied in [6].

In this work, we investigate further properties of the process X, solution of equation (2). It is well known (and this will be recalled hereafter) that the fBm B^H can be obtained by a transfer procedure from a classical Brownian motion W. Let $\{\bar{X}_t, t \in [0,T]\}$ be the solution of the following SDE on $\mathbb{R}^d$:

$$\bar{X}_t^i = x_0^i + \sum_{j=1}^{m} \int_0^t \sigma^{i,j}(\bar{X}_s)\mathrm{d}W_s + \int_0^t b^i(\bar{X}_s)\mathrm{d}s, \quad t \in [0,T],$$

$i = 1, \ldots, d$. The starting point of this work is the following remark. Assume for the moment that $m = d = 1$ and the function σ is identically equal to 1. A simple computation shows that

$$|X_t - \bar{X}_t| \leq |B_t^H - B_t| + c\int_0^t |B_s^H - W_s|\mathrm{e}^{c(t-s)}\mathrm{d}s$$

and using Lemma 3.2 in [5], we obtain that

$$\mathbb{E}\int_0^T |X_t - \bar{X}_t|^2\mathrm{d}t \leq c|H - 1/2|^2.$$

In this simple case, when H is close to $1/2$, the solution of the SDE (2) is close to the one driven by the Brownian motion W. It does not seem clear that analogous results hold when the diffusion coefficient is unspecified.

Our main result is that we can find a finite covering of the family of all the solutions of (2) (when coefficients b and σ belong to a certain class and parameter H varies in $[H_0, H_1]$ $(1/2 < H_0 \leq H_1 < 1)$) by $\mathbb{L}^2([0,T] \times \Omega)$ – open balls centred in solutions of equations driven by the Brownian motion from which our fBms are transferred.

We mainly use two ingredients. The first one is the relative compactness in $\mathbb{L}^2([0,T] \times \Omega)$ of this family of solutions and the second one is an approximation of the solution of SDEs constructed by replacing the driving processes B^H in the stochastic integrals by its polygonal approximation. In order to achieve this programme, we will need some estimations of the solution deterministic differential equations driven by the Hölder continuous functions of order greater than 1/2. These estimations are stated in Section 3 and they are more precise and accurate than the estimations contained in [9]: we give the precise behaviour (when H varies in $(1/2, 1)$) of the constants involved in our estimations.

In Section 2, we give the framework we use and state our results. The proofs of our main result are given in Section 4. Finally, some technical lemmas are proved in the Appendix.

2. Notations and main results

2.1 Notations

We briefly point out the well-known framework for an m-dimensional fBm B^H (see [13] for details). Let $\mathcal{H}$ be the Hilbert space defined as the closure of the set of step functions on

$[0, T]$ with values in $\mathbb{R}^m$ with respect to the scalar product

$$\left\langle \left(\mathbf{1}_{[0,t_1]}, \ldots, \mathbf{1}_{[0,t_m]}\right), \left(\mathbf{1}_{[0,s_1]}, \ldots, \mathbf{1}_{[0,s_m]}\right) \right\rangle_{\mathcal{H}} = \sum_{i=1}^{m} R_H(t_i, s_i).$$

We introduce the operator $\mathcal{K}_H^*$ defined for any $\varphi = (\varphi^1, \ldots, \varphi^m) \in \mathcal{H}$ and $i = 1, \ldots, m$ by

$$(\mathcal{K}_H^* \varphi^i)(s) = c_H s^{1/2-H} \int_s^T t^{H-1/2} (t-s)^{H-3/2} \varphi^i(t) \mathrm{d}t.$$

Thus, $\mathcal{K}_H^*$ provides an isometry between the Hilbert space $\mathcal{H}$ and a closed subspace of $L^2(0, T; \mathbb{R}^m)$ and it holds for any $\varphi, \psi \in \mathcal{H}$,

$$\langle \varphi, \psi \rangle_{\mathcal{H}} = \langle \mathcal{K}_H^* \varphi, \mathcal{K}_H^* \psi \rangle_{L^2(0,T;\mathbb{R}^m)} = \mathbb{E}\left(B^H(\varphi) B^H(\psi)\right).$$

The following inequality will be used in the following: there exists a constant $c(T)$ that depends only on T such that

$$\left\| \mathcal{K}_H^* \varphi \right\|_{\mathbb{L}^2([0,T])}^2 \leq c(T) \int_0^T \varphi^2(t) t^{1/2-H} \, \mathrm{d}t, \tag{3}$$

provided that the right-hand side of the above inequality is finite.

There is a link between the stochastic integration of deterministic integrand with respect to the fBm and with respect to a Wiener process which is naturally associated with B^H. This correspondence is usually called the transfer principle. The process $W = (W_t)_{t \in [0,T]}$ defined by

$$W_t = B^H\left((\mathcal{K}_H^*)^{-1} \left((\mathbf{1}_{[0,t]}, \ldots, \mathbf{1}_{[0,t]}) \right) \right)$$

is a Wiener process, and the process B^H has the integral representation

$$B_t^{H,i} = \int_0^t K_H(t,s) \mathrm{d}W_s^i, \quad i = 1, \ldots, m,$$

where the square integrable kernel K_H is defined for $s < t$ by

$$K_H(t,s) = c_H s^{1/2-H} \int_s^t (u-s)^{H-3/2} u^{H-1/2} \, \mathrm{d}u$$

with

$$c_H = \left(\frac{H(2H-1)}{\beta(2-2H, H-1/2)} \right)^{1/2}$$

(β denotes the Beta function). We set $K_H(t,s) = 0$ if $s \geq t$.

2.2 Main results

For $c_0, c_1, c_2 > 0$ we denote

- $\mathcal{C}_{c_0,c_1}^1$ the family of all differentiable functions $\varphi : \mathbb{R}^d \to \mathbb{R}$ such that

$$\begin{cases} \|\varphi\|_\infty & \leq c_0 \\ \|\varphi'\|_\infty & \leq c_1; \end{cases}$$

- $\mathcal{C}^2_{c_0,c_1,c_2}$ the family of all twice differentiable functions $\varphi : \mathbb{R}^d \to \mathbb{R}$ such that

$$\begin{cases} \|\varphi\|_\infty & \leq c_0 \\ \|\varphi'\|_\infty & \leq c_1 \\ \|\varphi''\|_\infty & \leq c_2. \end{cases}$$

We consider a family of fBms $(B^H)_{H\in[1/2,1)}$ defined on a complete filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$ transferred from a unique Brownian motion $(W_t)_{t\in[0,T]}$.

Let $H > 1/2$, $x_0 \in \mathbb{R}^d$, b^i and $\sigma^{i,j}$ belong, respectively, to $\mathcal{C}^1_{c_0,c_1}$ and $\mathcal{C}^2_{c_0,c_1,c_2}$ for $i = 1, \ldots, d$ and $j = 1, \ldots, m$, we denote $(X_t(H, x_0, b, \sigma))_{t\in[0,T]}$ the solution of equation (2)

$$X^i_t(H, x_0, b, \sigma) = x^i_0 + \sum_{j=1}^m \int_0^t \sigma^{i,j}(X_s(H, x_0, b, \sigma))\mathrm{d}B^{H,j}_s + \int_0^t b^i(X_s(H, x_0, b, \sigma))\mathrm{d}s.$$

The main result of this work deals with the set $\mathcal{S}^{H_0,H_1}_{c_0,c_1,c_2}$ of all the solutions of SDEs (2) driven by fraction Brownian motion B^H (when H varies in $[H_0, H_1]$ and the coefficients $b \in \mathcal{C}^1_{c_0,c_1}$ and $\sigma \in \mathcal{C}^2_{c_0,c_1,c_2}$), namely

$$\mathcal{S}^{H_0,H_1}_{c_0,c_1,c_2} = \Big\{(X_t(H, x_0, b, \sigma))_{t\in[0,T]} : H_0 \leq H \leq H_1, |x_0| \leq c_0, \\ b^i \in \mathcal{C}^1_{c_0,c_1}, \quad \sigma^{ij} \in \mathcal{C}^2_{c_0,c_1,c_2}, \ i = 1, \ldots, d, \ j = 1, \ldots, m\Big\}.$$

Our result states that we can find a finite covering of $\mathcal{S}^{H_0,H_1}_{c_0,c_1,c_2}$ by open balls of $\mathbb{L}^2([0, T] \times \Omega)$ centred in some solutions of SDEs driven by the Brownian motion W.

Theorem 1. Let $c_0, c_1, c_2 > 0$ and $1/2 < H_0 \leq H_1 < 1$. There exists $R > 0$, an integer N and for $k = 1, \ldots, N$ there exists:

- some initial values $x_{(k),0} \in \mathbb{R}^d$ such that $|x_{(k),0}| \leq c_0$,
- some drift coefficients $b^i_{(k)} \in \mathcal{C}^1_{c_0,c_1}$ for $i = 1, \ldots, d$,
- some diffusion coefficients $\sigma^{ij}_{(k)} \in \mathcal{C}^2_{c_0,c_1,c_2}$ for $i = 1, \ldots, d$ and $j = 1, \ldots, m$

such that the set

$$\mathcal{S}^{H_0,H_1}_{c_0,c_1,c_2} \subset \bigcup_{k=1}^N \Big\{Z \in \mathbb{L}^2([0, T] \times \Omega) : \|Z - \bar{X}_{(k)}\|_{\mathbb{L}^2([0,T]\times\Omega)} < R\Big\},$$

where the process $(\bar{X}_{(k),t})_{t\in[0,T]}$ is the unique solution of the SDE

$$\bar{X}^i_{(k),t} = x^i_{(k),0} + \sum_{j=1}^m \int_0^t \sigma^{i,j}_{(k)}(\bar{X}_{(k),s})\mathrm{d}W^j_s + \int_0^t b^i_{(k)}(\bar{X}_{(k),s})\mathrm{d}s.$$

We will make use of the following compactness property of Itô's functional. This property has its own interest.

Proposition 2. For any $c_0, c_1, c_2 > 0$ and $1/2 < H_0 \leq H_1 < 1$, the set $\mathcal{S}^{H_0,H_1}_{c_0,c_1,c_2}$ is relatively compact in $\mathbb{L}^2([0, T] \times \Omega)$.

The classical counterpart of this property for Brownian motion can be found in [4].

To prove these results, we will need estimations of the solution of (2) uniformly when the drift coefficient is in class $\mathcal{C}^1_{c_0,c_1}$ and when the diffusion coefficient is in $\mathcal{C}^2_{c_0,c_1,c_2}$. This is quite classical. The uniformity with respect to $H \in [H_0, H_1]$ of our estimations requires more technical computations and is less classical. This is the aim of the next section.

3. Deterministic differential equation driven by rough functions

First, we introduce some preliminaries. For $0 < \lambda \leq 1$ and $0 \leq a < b \leq T$, we denote by $C^\lambda(a,b;\mathbb{R}^d)$ the space of λ-Hölder continuous functions $f : [a,b] \to \mathbb{R}^d$, equipped with the norm

$$\|f\|_\lambda := \|f\|_{a,b,\infty} + \|f\|_{a,b,\lambda},$$

where

$$\|f\|_{a,b,\infty} = \sup_{a \leq r \leq b} |f(r)|,$$
$$\|f\|_{a,b,\lambda} = \sup_{a \leq r \leq s \leq b} \frac{|f(s) - f(r)|}{|s-r|^\lambda}.$$

We simply write $C^\lambda(a,b)$ when $d = 1$. Suppose that $f \in C^\lambda(a,b)$ and $g \in C^\mu(a,b)$ with $\lambda + \mu > 1$. From [17], the Riemann–Stieltjes integral $\int_a^b f\,\mathrm{d}g$ exists. In [18], the author provides an explicit expression for the integral $\int_a^b f\,\mathrm{d}g$ in terms of fractional derivatives. Let α be such that $\lambda > \alpha$ and $\beta > 1 - \alpha$. Then, the Riemann–Stieltjes integral can be expressed as

$$\int_a^b f_t\,\mathrm{d}g_t = (-1)^\alpha \int_a^b \left(D^\alpha_{a+}f\right)(t)\left(D^{1-\alpha}_{b-}g_{b-}\right)(t)\mathrm{d}t, \tag{4}$$

where

$$D^\alpha_{a+}f(t) = \frac{1}{\Gamma(1-\alpha)}\left(\frac{f(t)}{(t-a)^\alpha} + \alpha\int_a^t \frac{f(t)-f(s)}{(t-s)^{\alpha+1}}\mathrm{d}s\right),$$

and

$$D^\alpha_{b-}g_{b-}(t) = \frac{(-1)^\alpha}{\Gamma(1-\alpha)}\left(\frac{g(t)-g(b)}{(b-t)^\alpha} + \alpha\int_t^b \frac{g(t)-g(s)}{(s-t)^{\alpha+1}}\mathrm{d}s\right).$$

We refer to [16] for further details on fractional operators. Set $1/2 < \beta < 1$ and let $g \in C^\beta(0,T;\mathbb{R}^m)$. We shall work with deterministic differential equation on $\mathbb{R}^d$ of the form

$$x^i_t = x^i_0 + \int_0^t b^i(x_s)\mathrm{d}s + \sum_{j=1}^m \int_0^t \sigma^{i,j}(x_s)\mathrm{d}g^j_s, \quad t \in [0,T], \tag{5}$$

$i = 1, \ldots, d$, $x_0 \in \mathbb{R}^d$. We introduce the following assumptions.

H1: There exists $b_0, b_1 \in \mathbb{R}$ such that $b^i \in \mathcal{C}^1_{b_0,b_1}$, $1 \leq i \leq d$.

H2: There exists $c_0, c_1, c_2 \in \mathbb{R}$ such that $\sigma^{i,j} \in \mathcal{C}^2_{c_0,c_1,c_2}$, $1 \leq i \leq d$ and $1 \leq j \leq m$.

It is proved in [14] (Theorem 5.1) that if $1 - \beta < \alpha < 1/2$, the above equation has a unique $(1-\alpha)$-Hölder continuous solution. The estimates on the solution $(x_t)_{t\in[0,T]}$ obtained in [14] were improved in [9]. The following result is a more precise and accurate version of these estimates.

THEOREM 3. Let g be Hölder continuous of order $1/2 < \beta < 1$. Assume that b satisfies condition (H1) and σ satisfies (H2). Then for all T, there exists a constant $k(T)$ that depends only on T such that the solution $(x_t)_{t\in[0,T]}$ of equation (5) satisfies

$$\|x\|_\beta \leq |x_0| + k(T)(b_0 \vee c_0)(1+c_1^2)\left[1+\left(\frac{\|g\|_{0,T,\beta}}{\beta-1/2}\right)^{1+1/\beta}\right]. \tag{6}$$

Proof. We follow the arguments developed in [9]. We restrict ourselves to the case $d=m=1$ for simplicity.

First, we prove that

$$\|x\|_{0,T,\infty} \leq |x_0| + k(T)(b_0 \vee c_0)\left[1+\left(\frac{c_1\|g\|_{0,T,\beta}}{\beta-1/2}\right)^{1/\beta}\right]. \tag{7}$$

With $1-\beta < \alpha < 1/2$, we use (4) and we obtain for all $0 \leq s,t \leq T$

$$\left|\int_s^t \sigma(x_r)\mathrm{d}g_r\right| \leq \int_s^t |D_{s+}^{\alpha}\sigma(x_r)D_{t-}^{1-\alpha}g_{t-}(r)|\mathrm{d}r.$$

We have

$$\begin{aligned} |D_{t-}^{1-\alpha}g_{t-}(r)| &\leq \frac{\beta}{(\alpha+\beta-1)\Gamma(\alpha)}\|g\|_{0,T,\beta}|t-r|^{\alpha+\beta-1} \quad \text{and} \\ |D_{s+}^{\alpha}\sigma(x_r)| &\leq \frac{c_0}{\Gamma(1-\alpha)}(r-s)^{-\alpha} + \frac{\alpha c_1}{(\beta-\alpha)\Gamma(1-\alpha)}\|x\|_{s,r,\beta}(r-s)^{\beta-\alpha}. \end{aligned} \tag{8}$$

It follows that

$$\begin{aligned} \left|\int_s^t \sigma(x_r)\mathrm{d}g_r\right| \leq{}& \frac{c_0\beta}{(\alpha+\beta-1)\Gamma(\alpha)\Gamma(1-\alpha)}\|g\|_{0,T,\beta}\int_s^t (r-s)^{-\alpha}(t-r)^{\alpha+\beta-1}\,\mathrm{d}r \\ &+ \frac{\beta\alpha c_1}{(\beta-\alpha)(\alpha+\beta-1)\Gamma(\alpha)\Gamma(1-\alpha)}\|g\|_{0,T,\beta}\|x\|_{s,t,\beta} \\ &\int_s^t (r-s)^{\beta-\alpha}(t-r)^{\alpha+\beta-1}\,\mathrm{d}r. \end{aligned}$$

We recall that the Beta function is defined by $\mathcal{B}(a,b) = \int_0^1 (1-\xi)^{a-1}\xi^{b-1}d\xi = ((\Gamma(a)\Gamma(b))/\Gamma(a+b))$. The change of variables $r=(t-s)\xi+s$ implies

$$\begin{aligned} \left|\int_s^t \sigma(x_r)\mathrm{d}g_r\right| &\leq k_{\alpha,\beta}\|g\|_{0,T,\beta}\left[c_0(t-s)^\beta + c_1\|x\|_{s,t,\beta}(t-s)^{2\beta}\right] \quad \text{with} \\ k_{\alpha,\beta} &= \frac{\beta\mathcal{B}(\alpha+\beta,1-\alpha)}{(\alpha+\beta-1)\Gamma(\alpha)\Gamma(1-\alpha)} + \frac{\alpha\beta\mathcal{B}(\alpha+\beta,1+\beta-\alpha)}{(\alpha+\beta-1)(\beta-\alpha)\Gamma(\alpha)\Gamma(1-\alpha)}. \end{aligned} \tag{9}$$

We will prove hereafter that in fact $k_{\alpha,\beta} \leq \kappa/(\beta-1/2)$, where κ is a universal constant independent of H. For the moment, we write

$$|x_t - x_s| \leq (t-s)b_0 + k_{\alpha,\beta}\|g\|_{0,T,\beta}\left[c_0(t-s)^\beta + c_1\|x\|_{s,t,\beta}(t-s)^{2\beta}\right]$$

and if $|t-s|$ sufficiently small

$$\begin{aligned}\|x\|_{s,t,\beta} &\leq (t-s)^{1-\beta} b_0 + c_0 k_{\alpha,\beta}\|g\|_{0,T,\beta} + c_1 k_{\alpha,\beta}\|g\|_{0,T,\beta}\|x\|_{s,t,\beta}(t-s)^{\beta} \\ &\leq (T^{1-\beta} b_0 + c_0 k_{\alpha,\beta}\|g\|_{0,T,\beta})\big[1 - c_1 k_{\alpha,\beta}\|g\|_{0,T,\beta}(t-s)^{\beta}\big]^{-1}. \qquad (10)\end{aligned}$$

It follows that

$$\begin{aligned}|x_t| &\leq |x_s| + \|x\|_{s,t,\beta}(t-s)^{\beta} \\ &\leq |x_s| + \left(\left(T^{1-\beta} b_0 + c_0 k_{\alpha,\beta}\|g\|_{0,T,\beta}\right) \ \left[1 - c_1 k_{\alpha,\beta}\|g\|_{0,T,\beta}(t-s)^{\beta}\right]^{-1}\right)(t-s)^{\beta}.\end{aligned}$$

Let $B = T^{1-\beta} b_0 + c_0 k_{\alpha,\beta}\|g\|_{0,T,\beta}$ and $A = c_1 k_{\alpha,\beta}\|g\|_{0,T,\beta}$. We divide the interval of length $\Delta = ((1-\beta)/A)^{1/\beta}$ and we apply recursively the above inequality. Consequently, we have

$$\|x\|_{0,T,\infty} \leq |x_0| + \mathrm{TB}(1 - A\Delta^{\beta})^{-1}\Delta^{\beta-1} \leq |x_0| + \mathrm{TB}\frac{A^{(1-\beta)/\beta}}{\beta(1-\beta)^{(1-\beta)/\beta}}$$

and since for $1/2 < \beta < 1$, $\beta(1-\beta)^{(\beta-1)/\beta} \leq 2\mathrm{e}^{1/e} \leq 3$ we can write that

$$\|x\|_{0,T,\infty} \leq |x_0| + 3T(1+T)b_0\left(c_1 k_{\alpha,\beta}\|g\|_{0,T,\beta}\right)^{(1-\beta)/\beta} + 3Tc_0\left(c_1 k_{\alpha,\beta}\|g\|_{0,T,\beta}\right)^{1/\beta}.$$

But for $1/2 < \beta < 1$ and $x > 0$, $x^{1/\beta} + x^{1/\beta - 1} \leq 2(1 + x^{1/\beta})$ and we get

$$\|x\|_{0,T,\infty} \leq |x_0| + 6T(1+T)(b_0 \vee c_0)\left[1 + \left(c_1 k_{\alpha,\beta}\|g\|_{0,T,\beta}\right)^{1/\beta}\right]. \qquad (11)$$

Now we estimate $k_{\alpha,\beta}$ defined in (9). We rewrite $k_{\alpha,\beta}$ as

$$k_{\alpha,\beta} = \frac{\beta\Gamma(\alpha+\beta)}{(\alpha+\beta-1)\Gamma(\alpha)}\left(\frac{1}{\Gamma(1+\beta)} + \frac{\alpha\Gamma(1+\beta-\alpha)}{(\beta-\alpha)\Gamma(1-\alpha)\Gamma(1+2\beta)}\right).$$

We remark that

$$\begin{aligned}&1/2 \leq \alpha+\beta \leq 3/2, \quad 3/2 \leq 1+\beta \leq 2, \quad 1 \leq 1+\beta-\alpha \leq 2, \\ &1/2 \leq 1-\alpha \leq 1 \ \text{ and } \ 1 \leq 1+2\beta \leq 3.\end{aligned}$$

So if we denote $\Gamma_0 = \sup_{x\in[1/2,3]}\Gamma(x)$ and $\gamma_0 = \inf_{x\in[1/2,3]}\Gamma(x)$, using $\beta \leq 1$ implies

$$k_{\alpha,\beta} \leq \frac{\Gamma_0}{(\alpha+\beta-1)\Gamma(\alpha)}\left(\frac{1}{\gamma_0} + \frac{1/2\Gamma_0}{(\beta-\alpha)\gamma_0^2}\right).$$

It remains to treat the singularity when α is close to 0 (or equivalently β close to 1). First if $\alpha+\beta-1 > \alpha$, then

$$\frac{1}{(\alpha+\beta-1)\Gamma(\alpha)} \leq \frac{1}{\alpha\Gamma(\alpha)} \longrightarrow 1 \quad \text{as } \ \alpha \downarrow 0.$$

In the second case when $\alpha+\beta-1 < \alpha$, we use the fact that the function $x \longmapsto \Gamma(x)$ is decreasing for small values of $0 < x \leq 1$

$$\frac{1}{(\alpha+\beta-1)\Gamma(\alpha)} \leq \frac{1}{(\alpha+\beta-1)\Gamma(\alpha+\beta-1)} \longrightarrow 1 \quad \text{as } \ \alpha \downarrow 0.$$

Consequently, there exists a universal constant κ such that $k_{\alpha,\beta} \leq \kappa/(\beta-\alpha)$. Moreover, $\alpha < 1/2$ implies that $1/(\beta-\alpha) < 1/(\beta-1/2)$ so we have

$$k_{\alpha,\beta} \leq \frac{\kappa}{\beta - 1/2}. \tag{12}$$

Since $x \mapsto x^{1/\beta}$ is an increasing function, we deduce that

$$\left(\frac{\kappa}{\beta-\alpha}\right)^{1/\beta} \leq \frac{\kappa^2}{(\beta-1/2)^{1/\beta}}.$$

We report the above estimation into (11) and we get the existence of a constant that depends only on T such that (7) is satisfied.

Now we show that

$$\|x\|_{0,T,\beta} \leq k(T)(b_0 \vee c_0)c_1^{1/\beta}\left[1 + \left(\frac{\|g\|_{0,T,\beta}}{\beta-1/2}\right)^{1+1/\beta}\right]. \tag{13}$$

Set $\Delta = (2A)^{-1/\beta}$. On one hand, if $|t-s| \leq \Delta$, we use (10). On the other hand, when $|t-s| > \Delta$, we write

$$\begin{aligned}\frac{|x_t - x_s|}{|t-s|^\beta} &\leq \frac{|x_t - x_{t-\Delta}| + \cdots + |x_{s+\Delta} - x_s|}{|t-s|^\beta} \leq \frac{|x_t - x_{t-\Delta}|}{\Delta^\beta} + \cdots + \frac{|x_{s+\Delta} - x_s|}{\Delta^\beta}\\ &\leq \frac{|t-s|}{\Delta} B(1 - A\Delta^\beta)^{-1} \leq 2\mathrm{TB}(2A)^{1/\beta} \leq 8\mathrm{TBA}^{1/\beta}.\end{aligned}$$

We finally obtain that

$$\begin{aligned}\|x\|_{0,T,\beta} &\leq 8T(T^{1-\beta}b_0 + c_0 k_{\alpha,\beta}\|g\|_{0,T,\beta})\big(c_1 k_{\alpha,\beta}\|g\|_{0,T,\beta}\big)^{1/\beta}\\ &\leq 16T(1+T)(b_0 \vee c_0)c_1^{1/\beta}\Big(1 + (k_{\alpha,\beta}\|g\|_{0,T,\beta})^{1+1/\beta}\Big),\end{aligned}$$

and now we use (12) to obtain (13). Steps 1 and 2 imply (6). □

The next theorem gives the estimate for a linear deterministic equation. This result will be necessary to study the Malliavin regularity.

THEOREM 4. Let g be Hölder continuous of order $1/2 < \beta < 1$. Assume that b and σ satisfy, respectively, (H1) and (H2). Let x be the solution of (5) and for $i = 1, \ldots, d$, $j = 1, \ldots, m$, $0 \leq s \leq t \leq T$, consider $s \mapsto \Phi_t^{ij}(s)$ satisfying

$$\Phi_t^{ij}(s) = \sigma^{ij}(x_s) + \sum_{k=1}^{d}\int_s^t \partial_k b^i(x_u)\Phi_u^{k,j}(s)\mathrm{d}u + \sum_{k=1}^{d}\sum_{l=1}^{m}\int_s^t \partial_k \sigma^{il}(x_u)\Phi_u^{k,j}(s)\mathrm{d}g_u^l, \tag{14}$$

and $\Phi_t^{ij}(s) = 0$ if $s > t$.

For all T, there exists a constant $k(T)$ that depends only on T such that

$$\sup_{0\leq s\leq t\leq T} |\Phi_t(s)| \leq c_0 f_1\left(\frac{\|g\|_{0,T,\beta}}{\beta-1/2}\right), \tag{15}$$

$$\|\Phi_\cdot(s)\|_{s,T,\beta} \leq c_0 f_2\left(\frac{\|g\|_{0,T,\beta}}{\beta-1/2}\right), \tag{16}$$

with

$$f_1(z) = \exp\Big[k(T)((b_0 \vee b_1) + (c_0 \vee c_1 \vee c_2)z)^{1/\beta}\Big],$$
$$f_2(z) = k(T)\Big((b_0 \vee b_1)^{1/\beta} + ((c_0 \vee c_1 \vee c_2)z)^{1/\beta}\Big)f_1(z).$$

Proof. For simplicity we set $d = m = 1$. We refer to [15] for the existence and uniqueness of a solution of equation (14). We adapt the Proof of Theorem 3.2 in [9].

First we prove (15). We fix s and let $t' \geq t \geq s$. Instead of (8), we write

$$\begin{aligned}|D^{\alpha}_{s+}\sigma'(x_u)\Phi_u(s)| \leq \frac{c_1\|\Phi_{.}(s)\|_{s,u,\infty}}{\Gamma(1-\alpha)}(u-s)^{-\alpha} \\ + \frac{\alpha c_1}{(\beta-\alpha)\Gamma(1-\alpha)}\|\Phi_{.}(s)\|_{s,u,\beta}(u-s)^{\beta-\alpha} \\ + \frac{\alpha c_2}{(\beta-\alpha)\Gamma(1-\alpha)}\|\Phi_{.}(s)\|_{s,u,\infty}\|x\|_{s,u,\beta}(u-s)^{\beta-\alpha},\end{aligned}$$

and consequently

$$\begin{aligned}\|\Phi_{.}(s)\|_{t,t',\beta} &\leq (t'-t)^{1-\beta}b_1\|\Phi_{.}(s)\|_{t,t',\infty} + c_1k_{\alpha,\beta}\|g\|_{0,T,\beta}\|\Phi_{.}(s)\|_{t,t',\infty} \\ &\quad + c_2k_{\alpha,\beta}\|g\|_{0,T,\beta}\|x\|_{t,t',\beta}\|\Phi_{.}(s)\|_{t,t',\infty}(t'-t)^{\beta} \\ &\quad + c_1k_{\alpha,\beta}\|g\|_{0,T,\beta}\|\Phi_{.}(s)\|_{t,t',\beta}(t'-t)^{\beta}. \\ &\leq [((1+T)b_1 + c_1k_{\alpha,\beta}\|g\|_{0,T,\beta})\|\Phi_{.}(s)\|_{t,t',\infty} \\ &\quad + c_2k_{\alpha,\beta}\|g\|_{0,T,\beta}\|x\|_{t,t',\beta}\|\Phi_{.}(s)\|_{t,t',\infty}(t'-t)^{\beta}] \\ &\qquad \left(1 - c_1k_{\alpha,\beta}\|g\|_{0,T,\beta}(t'-t)^{\beta}\right)^{-1}.\end{aligned} \tag{17}$$

We divide the interval $[s, T]$ into $n = (T-s)/\Delta$ subintervals with $\Delta = t' - t$. We denote

$$A = (c_1 \vee c_2)k_{\alpha,\beta}\|g\|_{0,T,\beta},$$
$$B = (1+T)(b_0 \vee b_1) + (c_0 \vee c_1)k_{\alpha,\beta}\|g\|_{0,T,\beta}$$

and thanks to (10) we rewrite (17) as

$$\|\Phi_{.}(s)\|_{t,t',\beta} \leq \|\Phi_{.}(s)\|_{t,t',\infty}[B + A\Delta^{\beta}B(1-A\Delta^{\beta})^{-1}](1-A\Delta^{\beta})^{-1}\Delta^{\beta}. \tag{18}$$

We write

$$\|\Phi_{.}(s)\|_{t,t',\infty}\left[1 - [1 + A\Delta^{\beta}(1-A\Delta^{\beta})^{-1}](1-A\Delta^{\beta})^{-1}B\Delta^{\beta}\right] \leq |\Phi_t(s)|$$

and then

$$\sup_{s\leq r\leq t'}|\Phi_r(s)| \leq C\sup_{s\leq r\leq t}|\Phi_r(s)|, \quad \text{where}$$
$$C = \left[1 - (1 + A\Delta^{\beta}(1-A\Delta^{\beta})^{-1})(1-A\Delta^{\beta})^{-1}B\Delta^{\beta}\right]^{-1}.$$

Let $n = (T-s)/\Delta$. If we denote $Z_n(s) = \sup_{s\leq r\leq s+n\Delta}|\Phi_r(s)|$, we have

$$Z_n(s) \leq CZ_{n-1}(s) \leq \cdots \leq C^nZ_0(s) = C^n|\sigma(x_s)| \leq C^nc_0.$$

Consequently,

$$\sup_{s\leq t\leq T}|\Phi_t(s)| \leq c_0[1 - (1 + A\Delta^{\beta}(1-A\Delta^{\beta})^{-1})(1-A\Delta^{\beta})^{-1}B\Delta^{\beta}]^{-(T-s)/\Delta}.$$

If we choose $\Delta = (3(A \vee B))^{-1/\beta}$, then it satisfies $A\Delta^\beta \le 1/3$ and $B\Delta^\beta \le 1/3$. Hence,

$$\begin{aligned}\sup_{s\le t\le T} |\Phi_t(s)| &\le c_0 4^{(T-s)/\Delta}\\ &\le c_0 \exp\Big(4T\big[(1+T)(b_0 \vee b_1) + k_{\alpha,\beta}(c_0 \vee c_1 \vee c_2)\|g\|_{0,T,\beta}\big]^{1/\beta}\Big),\end{aligned}$$

and using estimations (12) of $k_{\alpha,\beta}$ we deduce (15).

We prove (16). On one hand, if $|t'-t| \ge \Delta$ we write

$$\begin{aligned}\frac{|\Phi_t(s)-\Phi_{t'}(s)|}{|t'-t|^\beta} &\le \frac{|\Phi_t(s)-\Phi_{t-\Delta}(s)| + \cdots + |\Phi_{t'+\Delta}(s)-\Phi_{t'}(s)|}{|t'-t|^\beta}\\ &\le \frac{|\Phi_t(s)-\Phi_{t-\Delta}(s)|}{\Delta^\beta} + \cdots + \frac{|\Phi_{t'+\Delta}(s)-\Phi_{t'}(s)|}{\Delta^\beta}.\end{aligned}$$

Then, we use (18) with $\Delta = (2(A \vee B))^{-1/\beta}$ and we obtain

$$\begin{aligned}\|\Phi_.(s)\|_{t,t',\beta} &\le \frac{|t'-t|}{\Delta}\|\Phi_.(s)\|_{t,t',\infty}[1 + A\Delta^\beta(1-A\Delta^\beta)^{-1}](1-A\Delta^\beta)^{-1}B\Delta^\beta\\ &\le k(T)(A \vee B)^{1/\beta}\|\Phi_.(s)\|_{t,t',\infty}.\end{aligned}$$

On the other hand, when $|t'-t| \le \Delta$, we use directly (18). In both cases, taking into account (15) yields (16). $\square$

4. Proof of Theorem 1

The results of the stochastic differential equation (2) are obtained almost surely using equation (5) when we replace g in (5) by the trajectories of the fBm. More precisely in [14] (Theorem 2.1), the authors proved that the stochastic process X exists almost surely and $t \mapsto X_t(\omega) \in C^{1-\alpha}(0,T;\mathbb{R}^d)$ with $1/2 > \alpha > 1-H$ and $1/2 < H < 1$. Theorem 3 will be useful in this context as soon as we have some estimation of the exponential moments of the Hölder norm of fBm's trajectories. This is the scope of the following classical Lemma (a self-contained proof is proposed in the Appendix).

Lemma 5. Let $1/2 < H_0 \le H < 1$, $H_0 - 1/2 > \varepsilon > 0$ and $p \ge 1/\varepsilon$, then

$$|B_t^{H,i} - B_s^{H,i}| \le \xi_{H,\varepsilon}|t-s|^{H-\varepsilon}, \quad i = 1, \ldots, m, \tag{19}$$

where $\xi_{H,\varepsilon}$ is a positive random variable such that

$$E\Big(\xi_{H,\varepsilon}^{2p}\Big) \le (16\sqrt{2}T)^{2p}\frac{(2p)!}{p!}. \tag{20}$$

Let $\kappa < 2$. For any constant c, there exists a constant $C_{\kappa,T}$ such that

$$\sup_{H_0\le H<1} \mathbb{E}\big[\exp\big(c\|B^H\|_{H-\varepsilon}^\kappa\big)\big] \le C_{\kappa,T}. \tag{21}$$

Now we introduce the following notations.

We denote by B^n the polygonal approximation of the fBm B^H and W^n the polygonal approximation of the Wiener process W. These approximations are defined by

$$B_t^n = \sum_{k=0}^{n-1}\left(B_{t_k}^H + \frac{n}{T}(t-t_k)\left(B_{t_k+1}^H - B_{t_k}^H\right)\right)\mathbf{1}_{(t_k,t_{k+1}]}(t),$$

$$W_t^n = \sum_{k=0}^{n-1}\left(W_{t_k} + \frac{n}{T}(t-t_k)(W_{t_k+1} - W_{t_k})\right)\mathbf{1}_{(t_k,t_{k+1}]}(t),$$

where $\{0 = t_0 < t_1 < \cdots < t_n = T\}$ is a uniform partition of the interval $[0, T]$ (that is $t_k = kT/n$ for $k = 0, \ldots, n$). As usual we will approximate the solution of SDEs thanks to these polygonal approximations when one replaces the driving processes in the stochastic integrals by its polygonal approximation. So we consider the following equations for $0 \le t \le T$, $i = 1, \ldots, d$ and $n \ge 1$:

$$X_t^{n,i} = x_0^i + \sum_{j=1}^{m}\int_0^t \sigma^{i,j}(X_s^n)\mathrm{d}B_s^{n,j} + \int_0^t b^i(X_s^n)\mathrm{d}s, \tag{22}$$

$$\bar{X}_t^{n,i} = x_0^i + \sum_{j=1}^{m}\int_0^t \sigma^{i,j}(\bar{X}_s^n)\mathrm{d}W_s^{n,j} + \int_0^t b^i(\bar{X}_s^n)\mathrm{d}s. \tag{23}$$

It is well known in the Brownian case (see [10]) that

$$\lim_{n\to\infty}\mathbb{E}\left(\sup_{0\le t\le T}\left|\bar{X}_t^n - \bar{X}_t\right|^2\right) = 0, \tag{24}$$

where we recall that the process $\bar{X}$ is the unique solution of

$$\bar{X}_t^i = x_0^i + \sum_{j=1}^{m}\int_0^t \sigma^{i,j}(\bar{X}_s)\mathrm{d}W_s^j + \int_0^t \tilde{b}^i(\bar{X}_s)\mathrm{d}s \quad \text{with}$$

$$\tilde{b}^i(x) = b^i(x) + \frac{1}{2}\sum_{j=1}^{d}\sum_{k=1}^{m}\left(\frac{\partial}{\partial x^j}\sigma^{i,k}(x)\right)\sigma^{j,k}(x). \tag{25}$$

An analogous convergence property holds true for the Wong-Zakaï type approximation of the SDE (32). Indeed, the almost-sure convergence is proved in [6]. Using (27) (in the following lemma), we will deduce by a dominated convergence argument that

$$\lim_{n\to\infty}\mathbb{E}\int_0^T\left|X_t - X_t^n\right|^2\mathrm{d}t = 0. \tag{26}$$

Lemma 6. If $(X_t^n)_{0\le t\le T}$ is the solution of the random ordinary differential equation (22), then

$$\sup_{n\ge1}\mathbb{E}\left(\sup_{0\le t\le T}|X_t^n|^2\right) \le c. \tag{27}$$

Proof. Let $1/2 < \beta < H$.

We prove that there exists a random variable $\xi_{H,\beta}$ such that

$$\sup_{H\in(1/2,1)}\mathbb{E}|\xi_{H,\beta}|^p < \infty \quad \text{and} \quad \sup_{n\ge1}\|B^n\|_{0,T,\beta} \le T^{H/2}\xi_{H,\beta}. \tag{28}$$

We set $0 < s < t < T$ and we assume that $t_l < s \le t_{l+1}$ and $t_k < t \le t_{k+1}$. On one hand, if $|t-s| \ge T/n$, then using (19) with $\varepsilon = (H-\beta)/2$ we have

$$\begin{aligned}
|B_t^n - B_s^n| &\le |B_{t_k} - B_{t_l}| + \left|\left(\frac{n}{T}\right)(t-t_k)(B_{t_{k+1}} - B_{t_k})\right| \\
&\quad + \left|\left(\frac{n}{T}\right)(t-t_l)(B_{t_{l+1}} - B_{t_l})\right| \\
&\le |B_{t_k} - B_{t_{l+1}}| + |B_{t_{l+1}} - B_{t_l}| + 2\xi_{H,\varepsilon}\left(\frac{T}{n}\right)^{H-\varepsilon} \\
&\le \xi_{H,\varepsilon}|t-s|^{H-\varepsilon} + 3\xi_{H,\varepsilon}(T/n)^{H-\varepsilon},
\end{aligned}$$

and then

$$\frac{|B_t^n - B_s^n|}{|t-s|^\beta} \le \xi_{H,\varepsilon}T^{H/2}\left(1 + 3n^{-H/2}\right) \le 4\xi_{H,\varepsilon}T^{H/2}. \tag{29}$$

On the other hand, if $|t-s| \le T/n$, then $t_k < s < t \le t_{k+1}$ or $t_{k-1} < s \le t_k < t \le t_{k+1}$. In any of these two cases, we have

$$\frac{|B_t^n - B_s^n|}{|t-s|^\beta} \le 3\xi_{H,\varepsilon}T^{H/2}. \tag{30}$$

Using (29) and (30) and the integrability property (20) of $\xi_{H,\varepsilon}$, we deduce (28).

Now, we apply Theorem 3 (precisely the estimation (7)) with $g = B^n$ and we obtain

$$\sup_{0\le t\le T}|X_t^n| \le C_{T,c_0} + C_{T,c_0}\left(\frac{c_1\|B^n\|_{0,T,\beta}}{(\beta - 1/2)}\right)^{1/\beta}. \tag{31}$$

□

Using (28), we deduce (27).

Now, the proof of Theorem 1 is simple.

Proof. We write

$$\|X_t - \bar{X}_t\|_{\mathbb{L}^2(\Omega)} = \|X_t - X_t^n\|_{\mathbb{L}^2(\Omega)} + \|X_t^n - \bar{X}_t^n\|_{\mathbb{L}^2(\Omega)} + \|\bar{X}_t^n - \bar{X}_t\|_{\mathbb{L}^2(\Omega)},$$

and then we use (24) and (26) in order to find $n_0 \ge 1$ such that

$$\|X - X^{n_0}\|_{\mathbb{L}^2([0,T]\times\Omega)} + \|\bar{X}^{n_0} - \bar{X}\|_{\mathbb{L}^2([0,T]\times\Omega)} \le 1.$$

Thanks to Lemma 6.7.2 in [10] and (27), we obtain the existence of a constant $C > 0$ that depends on T, H_0, c_0, c_1 and c_2 such that

$$\|X^{n_0} - \bar{X}^{n_0}\|_{\mathbb{L}^2([0,T]\times\Omega)} \le \|X^n\|_{\mathbb{L}^2([0,T]\times\Omega)} + \|\bar{X}^n\|_{\mathbb{L}^2([0,T]\times\Omega)} \le C.$$

Let $R = C + 1$. We define

$$\mathcal{S}^{1/2}_{c_0,c_1,c_2} = \Big\{(\bar{X}_t)_{t\in[0,T]} : \bar{X} \text{ is the solution of (25) with } |x_0| \le c_0, \\
b^i \in \mathcal{C}^1_{c_0,c_1},\ \ \sigma^{ij} \in \mathcal{C}^2_{c_0,c_1,c_2} i = 1,\dots,d;\ j = 1,\dots,m\Big\}.$$

It holds that

$$\mathcal{S}_{c_0,c_1,c_2}^{H_0,H_1} \subset \bigcup_{\bar{X}\in\mathcal{S}_{c_0,c_1,c_2}^{1/2}} \left\{ Z \in \mathbb{L}^2([0,T]\times\Omega) : \|Z-\bar{X}\|_{\mathbb{L}^2([0,T]\times\Omega)} < R \right\}$$

and the relative compactness of $\mathcal{S}_{c_0,c_1,c_2}^{H_0,H_1}$ in $\mathbb{L}^2([0,T]\times\Omega)$ (Proposition 2) yields Theorem 1. □

5. Proof of Proposition 2

To prove the relative compactness of the set $\mathcal{S}_{c_0,c_1,c_2}^{H_0,H_1}$, we use the relative compactness result proved in [1] (similar result can be found in [4,19]). These results state that this set is bounded in $\mathbb{L}^2([0,T]\times\Omega)$ and if we have the boundedness of the Malliavin derivative (and of their increments), then our set will be relatively compact (see [1], Theorem 2 for a precise statement).

First, we review the properties that we easily obtain, thanks to the previous technical results of this work. Any process $X \in \mathcal{S}_{c_0,c_1,c_2}^{H_0,H_1}$ satisfies for $i = 1, \ldots, d$

$$X_t^i = x_0^i + \sum_{j=1}^m \int_0^t \sigma^{i,j}(X_s)\mathrm{d}B_s^{H,j} + \int_0^t b^i(X_s)\mathrm{d}s, \quad t\in[0,T]. \tag{32}$$

Applying Theorem 3 and Lemma 5 will give the following properties:

$$\sup_{X\in\mathcal{S}_{c_0,c_1,c_2}^{H_0,H_1}} \|X\|_{\mathbb{L}^2([0,T]\times\Omega)} < \infty. \tag{33}$$

Indeed by Theorem 3

$$\mathbb{E}\int_0^T |X_t|^2\mathrm{d}t \le C_{T,c_0} + C_{T,c_0}\mathbb{E}\left[\left(\frac{c_1\|B^H\|_{0,T,\beta}}{(\beta-1/2)}\right)^{2/\beta}\right],$$

and by Lemma 5, $\|B^H\|_\beta$ has moments of all orders uniformly bounded with respect to H. Since $1/2 < H_0 \le H \le H_1 < 1$, we can choose $\beta = 1/2(H_0+1/2)$ and we have

$$\sup_{H_0\le H\le H_1} \mathbb{E}\int_0^T |X_t|^2\mathrm{d}t \le C_{T,c_0} + C_{T,c_0}\left(\frac{c_1 c(T)}{(2H_0-1)}\right)^4 \le C(T,c_0,c_1,H_0),$$

and consequently (33) holds. Using the same arguments, we obtain that for any $0 < a < b < T$ and $h\in\mathbb{R}$ (such that $|h| < \min(a, T-b)$)

$$\sup_{X\in\mathcal{S}_{c_0,c_1,c_2}^{H_0,H_1}} \int_a^b |\mathbb{E}X_{t+h} - \mathbb{E}X_t|^2\mathrm{d}t \to 0 \ \text{ as } \ |h|\to 0 \ \text{ and} \tag{34}$$

$$\sup_{X\in\mathcal{S}_{c_0,c_1,c_2}^{H_0,H_1}} \int_{[0,T]\setminus(a,b)} |\mathbb{E}X_t|^2\mathrm{d}t \to 0 \ \text{ when } \ a\downarrow 0 \ \text{ and } \ b\uparrow T. \tag{35}$$

We moreover need similar bounds in the Malliavin derivative, so we briefly recall the classical notation of Malliavin's calculus (see [13]). First, the derivative operator D^H is defined on the set of smooth cylindrical random variables $\mathcal{S}$ of the form

$$F = f\big(B^H(\varphi_1), \ldots, B^H(\varphi_n)\big), \tag{36}$$

where $n \geq 1, f \in C_b^\infty(\mathbb{R}^n)$ (f and all its partial derivatives are bounded), and $\varphi_i \in \mathcal{H}$. The derivative of a smooth cylindrical random variable F of the form (36) is the $\mathcal{H}$-valued random variable

$$D^H F := \sum_{i=1}^n \frac{\partial f}{\partial x_i}\left(B^H(\varphi_1), \ldots, B^H(\varphi_n)\right)\varphi_i.$$

This operator is closable from $L^p(\Omega)$ into $L^p(\Omega; \mathcal{H})$ for any $p \geq 1$ with respect to the norm

$$\|F\|_{1,p} := \left[\mathbb{E}(|F|^p) + \mathbb{E}\left(\|D^H F\|_{\mathcal{H}}^p\right)\right]^{1/p}.$$

There is a relation between the Malliavin calculus with respect to the fractional Brownian motion B^H and the Malliavin calculus with respect to the Brownian motion W. Denoting D the derivative operator with respect to W (and $D_W^{1,2}$ its domain) we have for any $F \in \mathbb{D}_W^{1,2} := \mathbb{D}^{1,2}$,

$$\mathcal{K}_H^*(D^H F) = DF. \tag{37}$$

Using the results of [15], the Malliavin derivative $D_s^j X_t^i$, $i = 1, \ldots, d, j = 1, \ldots, m$ of the process X exists and satisfies almost surely the equation

$$\begin{aligned} D_s^{H,j} X_t^i = \sigma^{ij}(X_s) &+ \sum_{k=1}^d \sum_{l=1}^m \int_s^t \partial_k \sigma^{il}(X_u) D_s^{H,j} X_u^k \mathrm{d}B_u^{H,l} \\ &+ \sum_{k=1}^d \int_s^t \partial_k b^i(X_u) D_s^{H,j} X_u^k \mathrm{d}u, \end{aligned}$$

if $s \leq t$ and 0 if $s > t$. We remark that we can apply Theorem 4 for the trajectories of the Malliavin derivative. Combined with Lemma 5 and inequality (3), we deduce that,

$$\sup_{X \in \mathcal{S}_{c_0,c_1,c_2}^{H_0,H_1}} \int_0^T \|X_t\|_{\mathbb{D}^{1,2}}^2 \mathrm{d}t < \infty. \tag{38}$$

Indeed we have

$$\begin{aligned} \int_0^T \int_0^T |D_\theta X_t|^2 \mathrm{d}\theta \mathrm{d}t &= \int_0^T \int_0^T |\mathcal{K}_H^*(D_\cdot^H X_t)(\theta)|^2 \mathrm{d}\theta \mathrm{d}t \\ &\leq c(T) \int_0^T \int_0^T |D_\theta^H X_t|^2 \theta^{1/2-H} \mathrm{d}\theta \mathrm{d}t, \end{aligned}$$

and thanks to (15) from Theorem 4 and (21) from Lemma 5 we obtain (38). In the same way we have

$$\sup_{X \in X \in \mathcal{S}_{c_0,c_1,c_2}^{H_0,H_1}} \mathbb{E} \int_{[0,T]^2 \setminus (a,b) \times (a',b')} |D_\theta X_t|^2 \mathrm{d}\theta \mathrm{d}t \to 0 \text{ when } a, a' \downarrow 0 \text{ and } b, b' \uparrow T. \tag{39}$$

In order to have the relative compactness property, it remains to prove that for any $0 < a < b < T$, $0 < a' < b' < T$ and $h, h' \in \mathbb{R}$ (such that $|h| \vee |h'| <$ $\min(a, a', T - b, T - b')$), it holds

$$\sup_{X \in \mathcal{S}_{c_0,c_1,c_2}^{H_0,H_1}} \mathbb{E} \int_a^b \int_{a'}^{b'} |D_{\theta+h} X_{t+h'} - D_\theta X_t|^2 \mathrm{d}\theta \mathrm{d}t \to 0 \text{ when } |h| \to 0 \text{ and } |h'| \to 0. \tag{40}$$

The proof is a little bit more technical. We write

$$\mathbb{E}\int_a^b\int_{a'}^{b'}|D_{\theta+h}X_{t+h'} - D_\theta X_t|^2\mathrm{d}\theta\mathrm{d}t \leq I_1(h') + I_2(h)$$

with

$$\begin{aligned}
I_1(h') &= \mathbb{E}\int_a^b\int_{a'}^{b'}|D_{\theta+h}X_{t+h'} - D_{\theta+h}X_t|^2\mathrm{d}\theta\mathrm{d}t \\
&= \mathbb{E}\int_a^b\int_{a'}^{b'}|(\mathcal{K}_H^* D_\cdot^H X_{t+h'})(\theta+h) - (\mathcal{K}_H^* D_\cdot^H X_t)(\theta+h)|^2\mathrm{d}\theta\mathrm{d}t, \\
I_2(h) &= \mathbb{E}\int_a^b\int_{a'}^{b'}|D_{\theta+h}X_t - D_\theta X_t|^2\mathrm{d}\theta\mathrm{d}t \\
&= \mathbb{E}\int_a^b\int_{a'}^{b'}|(\mathcal{K}_H^* D_\cdot^H X_t)(\theta+h) - (\mathcal{K}_H^* D_\cdot^H X_t)(\theta)|^2\mathrm{d}\theta\mathrm{d}t.
\end{aligned}$$

Using (3) and (16) in Theorem 4

$$\begin{aligned}
I_1(h') &\leq c(T)\mathbb{E}\int_a^b\int_{a'}^{b'}|D_{\theta+h}^H X_{t+h'} - D_{\theta+h}^H X_t|^2\theta^{1/2-H}\mathrm{d}\theta\mathrm{d}t \\
&\leq C(T, c_0)\left(\int_a^b\int_{a'}^{b'}\theta^{1/2-H}\mathrm{d}\theta\mathrm{d}t\right)\mathbb{E}\left[f_2\left(\frac{\|B^H\|_{0,T,\beta}}{\beta-1/2}\right)\right]^2|h'|^\beta
\end{aligned}$$

and the uniform convergence follows from Lemma 5. We will obtain the expected convergence result for the term $I_2(h)$ using the following inequality: for $\varepsilon < (H_0 - 1/2) \wedge 1/4$

$$I_2(h) \leq \frac{c_T h^{2\varepsilon}}{\varepsilon^2(H_0 - 1/2 - \varepsilon)}\int_a^b\int_{a'}^{b'}(1 + t^{1/2-H-2\varepsilon})|D_{\theta+h}X_t|^2\mathrm{d}\theta\mathrm{d}t \tag{41}$$

and the claim follows from (16) and Lemma 5. The proof of (41) is easy but tedious, so it is postponed in the Appendix.

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Appendix

Proof of Lemma 5

Proof. Although the proof of (19) is classical, for reading convenience we include it. With $\psi(u) = u^{2/\varepsilon}$ and $p(u) = u^H$ in Lemma 1.1 of [8], the Garsia–Rodemich–Rumsey inequality reads as follows:

$$|B_t^{H,i} - B_s^{H,i}| \leq 8\int_0^{|t-s|} \left(\frac{4\Delta}{u^2}\right)^{\varepsilon/2} Hu^{H-1}\mathrm{d}u,$$

where the random variable Δ is

$$\Delta = \int_0^T\int_0^T \frac{|B_t^{H,i} - B_s^{H,i}|^{2/\varepsilon}}{|t-s|^{2H/\varepsilon}}\mathrm{d}t\mathrm{d}s.$$

Using $H - \varepsilon > 1/2$, we have

$$\begin{aligned}|B_t^{H,i} - B_s^{H,i}| &\leq 8\,4^{\varepsilon/2}\Delta^{\varepsilon/2}\int_0^{|t-s|} H\,u^{H-1-\varepsilon}\mathrm{d}u \\ &\leq 8\,4^{\varepsilon/2}\Delta^{\varepsilon/2}\frac{H}{H-\varepsilon}|t-s|^{H-\varepsilon} \\ &\leq 8\,4^{\varepsilon/2}\Delta^{\varepsilon/2}|t-s|^{H-\varepsilon}.\end{aligned}$$

We denote $\xi_{H,\varepsilon} = 8\,4^{\varepsilon/2}\Delta^{\varepsilon/2}$ and for $p \geq 1/\varepsilon$ it holds

$$\begin{aligned}\mathbb{E}\xi_{H,\varepsilon}^{2p} &\leq 8^{2p}4^{p\varepsilon}\mathbb{E}\left(\int_0^T\int_0^T \frac{|B_t^{H,i} - B_s^{H,i}|^{2/\varepsilon}}{|t-s|^{2H/\varepsilon}}\mathrm{d}t\mathrm{d}s\right)^{p\varepsilon} \\ &\leq 8^{2p}4^{p\varepsilon}T^{2p\varepsilon}\int_0^T\int_0^T \frac{\mathbb{E}|B_t^{H,i} - B_s^{H,i}|^{2p}}{|t-s|^{2pH}}\frac{\mathrm{d}t\mathrm{d}s}{T^2} \\ &\leq 8^{2p}4^{p\varepsilon}T^{2p\varepsilon}\frac{(2p)!}{2^p p!} \leq (16\sqrt{2}T)^{2p}\frac{(2p)!}{p!}.\end{aligned}$$

Thus (19) and (20) are proved.

The last part of the lemma can be deduced tediously from Theorem 1.3.2 in [7] but in order to have uniform estimate with respect to H, it seems easier to make the following direct computations. Let ε be such that $\beta = H - \varepsilon$. Using (19) and (20), we have

$$\begin{aligned}\mathbb{E}\Big(\exp(\alpha\|B^H\|_\beta^2)\Big) &\leq \mathbb{E}\Big(\exp(\alpha\xi_{H,\beta}^2)\Big) \\ &\leq \mathbb{E}\left(\sum_{p=0}^{\infty}\frac{\alpha^p\,\xi_{H,\beta}^{2p}}{p!}\right) \\ &\leq \sum_{p=0}^{\infty}\frac{\alpha^p(16\sqrt{2}T)^{2p}(2p)!}{(p!)^2},\end{aligned}$$

and the right-hand side of the above inequality is finite provided that $\alpha \leq 2048T^2$. Anyway in this case it holds that

$$\sup_{H_0\leq H\leq 1}\mathbb{E}\Big(\exp(\alpha\|B^H\|_\beta^2)\Big) \leq c_T.$$

Applying Young's inequality implies (21). □

Proof of (41)

For convenience we denote $s \mapsto \varphi(s) := D_sX_t$ and we restrict ourselves to the case $h > 0$. We write for $a \leq t \leq b$

$$\begin{aligned}(\mathcal{K}_H^*\varphi)(t+h) - (\mathcal{K}_H^*\varphi)(t) = &c_H(t+h)^{1/2-H}\int_{t+h}^T r^{H-1/2}(r-t-h)^{H-3/2}\varphi(r)\mathrm{d}r \\ &- c_H t^{1/2-H}\int_t^T r^{H-1/2}(r-t)^{H-3/2}\varphi(r)\mathrm{d}r\end{aligned}$$

and then

$$\left|(\mathcal{K}_H^*\varphi)(t+h) - (\mathcal{K}_H^*\varphi)(t)\right| \leq I_1(h) + I_2(h) + I_3(h)$$

with

$$I_1(t,h) = c_H\Big(t^{1/2-H} - (t+h)^{1/2-H}\Big)\int_{t+h}^{T} r^{H-1/2}(r-t-h)^{H-3/2}|\varphi(r)|\mathrm{d}r,$$
$$I_2(t,h) = c_H t^{1/2-H}\int_{t}^{t+h} r^{H-1/2}(r-t)^{H-3/2}|\varphi(r)|\mathrm{d}r,$$
$$I_3(t,h) = c_H t^{1/2-H}\int_{t+h}^{T} r^{H-1/2}\big|(r-t-h)^{H-3/2} - (r-t)^{H-3/2}\big||\varphi(r)|\mathrm{d}r.$$

Let $\varepsilon > 0$ be such that $\varepsilon < (H-1/2)\wedge 1/4$. We use the inequality

$$s^{1/2-H} - (s+h)^{1/2-H} \leq \left(\frac{H-1/2}{\varepsilon}\right)h^{\varepsilon}s^{1/2-H-\varepsilon}$$

and the Jensen inequality

$$\begin{aligned}\int_a^b |I_1(t,h)|^2\mathrm{d}t \leq c_H^2\frac{(H-1/2)^2}{\varepsilon^2}h^{2\varepsilon}\int_a^b t^{1-2H-2\varepsilon}\frac{(T-t-h)^{2H-1}}{(H-1/2)^2}\\ \left(\int_{t+h}^{T} r^{2H-1}(r-t-h)^{H-3/2}|\varphi(r)|^2\mathrm{d}r\right)\mathrm{d}t\\ \leq c_T c_H^2\frac{h^{2\varepsilon}}{\varepsilon^2}\int_a^b t^{1-2H-2\varepsilon}\int_{t+h}^{T}(r-t-h)^{H-3/2}|\varphi(r)|^2\mathrm{d}r\mathrm{d}t\\ \leq c_T c_H^2\frac{h^{2\varepsilon}}{\varepsilon^2}\int_{a+h}^{T}\int_a^{r-h} t^{1-2H-2\varepsilon}(r-t-h)^{H-3/2}\mathrm{d}t|\varphi(r)|^2\mathrm{d}r\\ \leq c_T c_H^2\frac{h^{2\varepsilon}}{\varepsilon^2}\int_{a+h}^{T}\int_0^{r-h} t^{1-2H-2\varepsilon}(r-t-h)^{H-3/2}\mathrm{d}t|\varphi(r)|^2\mathrm{d}r,\end{aligned}$$

and the change of variables $\xi = t/(r-h)$ implies that

$$\begin{aligned}\int_a^b |I_1(t,h)|^2\mathrm{d}t \leq c_T c_H^2\mathcal{B}(2-2H-2\varepsilon; H-1/2)\\ \frac{h^{2\varepsilon}}{\varepsilon^2}\int_{a+h}^{T}(r-h)^{1/2-H-2\varepsilon}|\varphi(r)|^2\mathrm{d}r.\end{aligned}$$

Since $\sup_{H\in(1/2,1)}c_H^2\mathcal{B}(2-2H-2\varepsilon; H-1/2) < \infty$ and $(r-h)^{1/2-H-2\varepsilon} \leq 1 + r^{1/2-H-2\varepsilon}$ we have

$$\int_a^b |I_1(t,h)|^2\mathrm{d}t \leq c_T\frac{h^{2\varepsilon}}{\varepsilon^2}\int_h^T (r-h)^{1/2-H-2\varepsilon}|\varphi(r)|^2\mathrm{d}r. \tag{A1}$$

Analogously, we have

$$\begin{aligned}
\int_a^b |I_2(t,h)|^2 \mathrm{d}t &\leq c_H^2 \frac{h^{2H-1}}{(H-1/2)^2} \int_a^b t^{1-2H} \int_t^{t+h} r^{2H-1}(r-t)^{H-3/2} |\varphi(r)|^2 \mathrm{d}r\mathrm{d}t \\
&\leq c_T c_H^2 \frac{h^{2H-1}}{(H-1/2)^2} \int_a^b t^{1-2H} \int_t^{t+h} (r-t)^{H-3/2} |\varphi(r)|^2 \mathrm{d}r\mathrm{d}t \\
&\leq c_T c_H^2 \frac{h^{2H-1}}{(H-1/2)^2} \int_a^{b+h} |\varphi(r)|^2 \int_{r-h}^{r} t^{1-2H}(r-t)^{H-3/2} \mathrm{d}t\mathrm{d}r \\
&\leq c_T c_H^2 \frac{h^{2H-1}}{(H-1/2)^2} \int_a^{b+h} |\varphi(r)|^2 \int_0^{r} t^{1-2H}(r-t)^{H-3/2} \mathrm{d}t\mathrm{d}r \\
&\leq c_T c_H^2 \frac{h^{2H-1}}{(H-1/2)^2} \mathcal{B}(2-2H; H-1/2) \int_a^{b+h} |\varphi(r)|^2 r^{1/2-H} \mathrm{d}r,
\end{aligned}$$

and since $r^{1/2-H} \leq 1 + r^{1/2-H-2\varepsilon}$ we deduce that

$$\int_a^b |I_2(t,h)|^2 \mathrm{d}t \leq c_T \frac{h^{2H-1}}{H-1/2} \int_0^T |\varphi(r)|^2 r^{1/2-H} \mathrm{d}r. \tag{A2}$$

Thanks to the inequality $s^{1/2-H} - (s+h)^{1/2-H} \leq \big((H-1/2)/\varepsilon\big) h^{\varepsilon} s^{1/2-H-\varepsilon}$

$$\begin{aligned}
\int_a^b |I_3(t,h)|^2 \mathrm{d}t &\leq c_T c_H^2 \frac{h^{2\varepsilon}}{\varepsilon^2} \int_a^b t^{1-2H} \left(\int_{t+h}^T (r-t-h)^{H-3/2-\varepsilon} |\varphi(r)| \mathrm{d}r \right)^2 \mathrm{d}t \\
&\leq c_T c_H^2 \frac{h^{2\varepsilon}}{\varepsilon^2} \frac{(T-t-h)^{H-1/2-\varepsilon}}{H-1/2-\varepsilon} \int_a^b t^{1-2H} \left(\int_{t+h}^T (r-t-h)^{H-3/2-\varepsilon} |\varphi(r)|^2 \mathrm{d}r \right) \mathrm{d}t \\
&\leq \frac{c_T c_H^2 h^{2\varepsilon}}{\varepsilon^2 (H-1/2-\varepsilon)} \int_{a+h}^T |\varphi(r)|^2 \left(\int_a^{r-h} t^{1-2H}(r-t-h)^{H-3/2-\varepsilon} \mathrm{d}t \right) \mathrm{d}r \\
&\leq \frac{c_T c_H^2 h^{2\varepsilon} \mathcal{B}(2-2H; H-1/2-\varepsilon)}{\varepsilon^2 (H-1/2-\varepsilon)} \int_{a+h}^T |\varphi(r)|^2 (r-h)^{1/2-H-\varepsilon} \mathrm{d}r.
\end{aligned}$$

From $x\Gamma(x) \to 1$ as $x \to 0$ and $(r-h)^{1/2-H-\varepsilon} \leq 1 + r^{1/2-H-\varepsilon} \leq 2 + r^{1/2-H-2\varepsilon}$, we deduce

$$\int_a^b |I_3(t,h)|^2 \mathrm{d}t \leq c_T \frac{h^{2\varepsilon}}{\varepsilon^2} \frac{1}{(H-1/2-\varepsilon)^2} \int_h^T |\varphi(r)|^2 (r-h)^{1/2-H-\varepsilon} \mathrm{d}r. \tag{A3}$$

The estimations (A1), (A2) and (A3) imply (41).