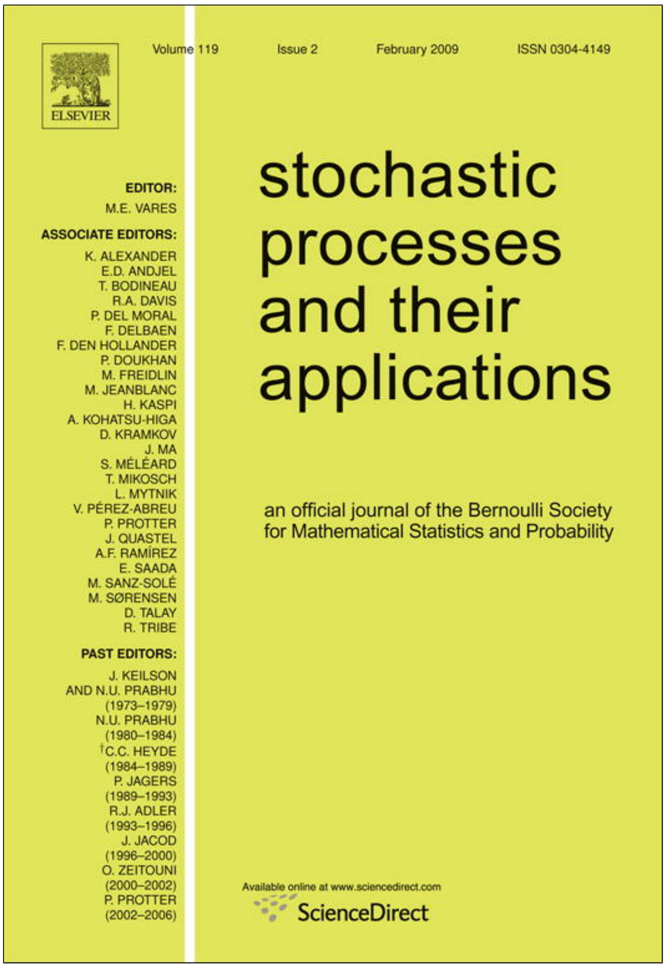
Volume 119
Issue 2
February 2009
ISSN 0304-4149
ELSEVIER
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Stochastic Processes and their Applications 119 (2009) 391–409
stochastic processes and their applications
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Malliavin calculus for stochastic differential equations driven by a fractional Brownian motion

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Received 1 March 2005; received in revised form 25 October 2006; accepted 4 February 2008
Available online 1 April 2008

Abstract

We prove the Malliavin regularity of the solution of a stochastic differential equation driven by a fractional Brownian motion of Hurst parameter $H > 0.5$. The result is based on the Fréchet differentiability with respect to the input function for deterministic differential equations driven by Hölder continuous functions. It is also shown that the law of the solution has a density with respect to the Lebesgue measure, under a suitable nondegeneracy condition.

Keywords: Stochastic differential equation; Malliavin calculus; Fractional Brownian motion

1. Introduction

Let $B = \{B_t, t \geq 0\}$ be an m-dimensional fractional Brownian motion (fBm for short) of Hurst parameter $H \in (0, 1)$. That is, B is a centered Gaussian process with the covariance function $\mathbb{E}(B_s^i B_t^j) = R_H(s,t)\delta_{ij}$, where

$$R_H(s,t) = \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right). \tag{1}$$

If $H = \frac{1}{2}$, B is a Brownian motion. From (1), it follows that $\mathbb{E}\left(|B_t - B_s|^2\right) = m|t-s|^H$ so the process B has α-Hölder continuous paths for all $\alpha \in (0, H)$. We refer the reader to [13] and

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0304-4149/$ - see front matter
doi:10.1016/j.spa.2008.02.016

references therein for further information about fBm and stochastic integration with respect to this process.

In this article we fix $\frac{1}{2} < H < 1$ and we consider the solution $\{X_t, t \in [0,T]\}$ of the following stochastic differential equation on $\mathbb{R}^d$:

$$X_t^i = x_0^i + \sum_{j=1}^m \int_0^t \sigma^{ij}(X_s)\mathrm{d}B_s^j + \int_0^t b^i(X_s)\mathrm{d}s, \quad t \in [0,T], \tag{2}$$

$i = 1, \ldots, d$, where $x_0 \in \mathbb{R}^d$ is the initial value of the process X.

The stochastic integral in (2) is a pathwise Riemann–Stieltjes integral (see Young [16]). Suppose that σ has bounded partial derivatives which are Hölder continuous of order $\lambda > \frac{1}{H} - 1$, and b is Lipschitz; then there is a unique solution to Eq. (2) which has Hölder continuous trajectories of order $H - \varepsilon$, for any $\varepsilon > 0$. This result has been proved by Lyons in [8] in the case $b = 0$, using the p-variation norm. The theory of rough paths analysis introduced by Lyons in [9] was used by Coutin and Qian to prove an existence and uniqueness result for Eq. (2) in the case $H \in (\frac{1}{4}, \frac{1}{2})$ (see [4]).

The Riemann–Stieltjes integral appearing in Eq. (2) can be expressed as a Lebesgue integral using a fractional integration by parts formula (see Zähle [17]). Using this formula for the Riemann–Stieltjes integral, Nualart and Răşcanu have established in [14] the existence of a unique solution for a class of general differential equations that includes (2).

The main purpose of our work is to study the regularity of the solution to Eq. (2) in the sense of Malliavin calculus, and to show the absolute continuity for the law of X_t for $t > 0$, assuming an ellipticity condition on the coefficient σ. First we establish a general result on the regularity with respect to the driven function for the solution of deterministic equations, using the techniques of fractional calculus developed in [14]. This allows us to deduce the differentiability of the solution to Eq. (2) in the direction of the Cameron–Martin space. These results are related to those proved by Lyons and Dong Li in [10] on the smoothness of Itô maps for such equations in terms of Fréchet–Gâteaux differentiability.

The regularity results obtained here have been used in a recent paper by Baudoin and Hairer [1] to show the smoothness of the density under a hypoellipticity Hörmander condition. This result requires also the existence of moments for the iterated derivatives, which has been established in [6]. In [11], the existence of a density for the solution of a one-dimensional equation is shown. See also [2,3] for other recent results.

The paper is organized as follows. In Section 2 we establish the Fréchet differentiability with respect to the input function for deterministic differential equations driven by Hölder continuous functions. Section 3 is devoted to analyzing stochastic differential equations driven by a fBm with Hurst parameter $H \in (\frac{1}{2}, 1)$, the main result being the differentiability of the solution in the directions of the Cameron–Martin space. In Section 4 we prove the absolute continuity of the solution under ellipticity assumptions. The proofs of some technical results are given in the Appendix.

2. Deterministic differential equations driven by rough functions

We first introduce some preliminaries. Given a measurable function $f : [0,T] \to \mathbb{R}^d$ and $\alpha \in (0, \frac{1}{2})$, we will make use of the notation

$$\Delta_t^\alpha(f) = |f(t)| + \int_0^t \frac{|f(t) - f(s)|}{|t-s|^{\alpha+1}}\mathrm{d}s.$$

We denote by $W_1^{\alpha}(0,T;\mathbb{R}^d)$ the space of measurable functions $f : [0,T] \to \mathbb{R}^d$ such that

$$\|f\|_{\alpha,1} := \sup_{t\in[0,T]} \Delta_t^{\alpha}(f) < \infty.$$

For any $0 < \lambda \leq 1$, denote by $C^{\lambda}(0,T;\mathbb{R}^d)$ the space of λ-Hölder continuous functions $f : [0,T] \to \mathbb{R}^d$, equipped with the norm

$$\|f\|_{\lambda} := \|f\|_{\infty} + \sup_{0\leq s<t\leq T} \frac{|f(t)-f(s)|}{(t-s)^{\lambda}},$$

where $\|f\|_{\infty} := \sup_{t\in[0,T]} |f(t)|$. We denote by $W_2^{1-\alpha}(0,T;\mathbb{R}^m)$ the space of measurable functions $g : [0,T] \to \mathbb{R}^m$ such that

$$\|g\|_{1-\alpha,2} := \sup_{0\leq s<t\leq T} \left(\frac{|g(t)-g(s)|}{(t-s)^{1-\alpha}} + \int_s^t \frac{|g(y)-g(s)|}{(y-s)^{2-\alpha}} \mathrm{d}y \right) < \infty.$$

Clearly for any $\varepsilon > 0$ such that $1-\alpha+\varepsilon \leq 1$ we have

$$C^{\alpha+\varepsilon}(0,T;\mathbb{R}^d) \subset W_1^{\alpha}(0,T;\mathbb{R}^d)$$

and

$$C^{1-\alpha+\varepsilon}(0,T;\mathbb{R}^m) \subset W_2^{1-\alpha}(0,T;\mathbb{R}^m) \subset C^{1-\alpha}(0,T;\mathbb{R}^m).$$

For $d = m = 1$ we simply write $W_1^{\alpha}(0,T)$, $C^{\lambda}(0,T)$, and $W_2^{1-\alpha}(0,T)$.

Suppose that $g \in W_2^{1-\alpha}(0,T)$ and $f \in W_1^{\alpha}(0,T)$. In [17], Zähle introduced the generalized Stieltjes integral

$$\int_0^T f_t \mathrm{d}g_t = (-1)^{\alpha} \int_0^T \left(D_{0+}^{\alpha} f\right)(t) \left(D_{T-}^{1-\alpha} g_{T-}\right)(t)\mathrm{d}t, \tag{3}$$

defined in terms of the fractional derivative operators

$$D_{0+}^{\alpha} f(x) = \frac{1}{\Gamma(1-\alpha)} \left(\frac{f(x)}{x^{\alpha}} + \alpha \int_0^x \frac{f(x)-f(y)}{(x-y)^{\alpha+1}} \mathrm{d}y \right),$$

and

$$D_{T-}^{\alpha} g_{T-}(x) = \frac{(-1)^{\alpha}}{\Gamma(1-\alpha)} \left(\frac{g(x)-g(T)}{(T-x)^{\alpha}} + \alpha \int_x^T \frac{g(x)-g(y)}{(x-y)^{\alpha+1}} \mathrm{d}y \right).$$

We refer the reader to [15] for further details on fractional operators. Zähle proved that if $f \in C^{\alpha+\varepsilon}(0,T)$, then this integral coincides with the Riemann–Stieltjes integral, which exists by the results of Young (see [16]). Using formula (3), Nualart and Răşcanu have derived the following estimates (see [14], Propositions 4.1 and 4.3).

Proposition 1. *Fix $0 < \alpha < \frac{1}{2}$. Given two functions $g \in W_2^{1-\alpha}(0,T)$ and $f \in W_1^{\alpha}(0,T)$, we define $G_t(f) = \int_0^t f_s \mathrm{d}g_s$ and $F_t(f) = \int_0^t f_s \mathrm{d}s$.*

(i) *The function $G(f)$ belongs to $C^{1-\alpha}(0,T)$ and we have*

$$\Delta_t^{\alpha}(G(f)) \leq c_{\alpha,T} \|g\|_{1-\alpha,2} \int_0^t \left[(t-r)^{-2\alpha} + t^{-\alpha}\right] \Delta_r^{\alpha}(f)\mathrm{d}r, \tag{4}$$

$$\|G(f)\|_{1-\alpha} \leq c_{\alpha,T} \|g\|_{1-\alpha,2} \|f\|_{\alpha,1}, \tag{5}$$

with a constant $c_{\alpha,T}$ which depends only on α and T.

(ii) *The function $F(f)$ belongs to $C^1(0,T)$ and moreover*

$$\Delta_t^\alpha(F(f)) \le c_{\alpha,T} \int_0^t \frac{|f_s|}{(t-s)^\alpha} \mathrm{d}s, \tag{6}$$

$$\|F(f)\|_1 \quad \le c_T \|f\|_\infty, \tag{7}$$

with a constant $c_{\alpha,T}$ which depends only on α and T.

We first study deterministic differential equations driven by Hölder continuous functions of order strictly larger that $\frac{1}{2}$. Fix $0 < \alpha < \frac{1}{2}$. Let $g \in W_2^{1-\alpha}(0,T;\mathbb{R}^m)$ and consider the deterministic differential equation on $\mathbb{R}^d$

$$x_t^i = x_0^i + \int_0^t b^i(x_s)\mathrm{d}s + \sum_{j=1}^m \int_0^t \sigma^{ij}(x_s)\mathrm{d}g_s^j\,, \quad t \in [0,T]\,, \tag{8}$$

$i = 1,\ldots,d$, where $x_0 \in \mathbb{R}^d$.

For any integer $k \ge 1$ we denote by C_b^k the class of real-valued functions on $\mathbb{R}^d$ which are k times continuously differentiable with bounded partial derivatives up to the kth order. We also denote by C_b^∞ the class of infinitely differentiable functions on $\mathbb{R}^d$ with bounded partial derivatives of all orders.

In [14], the authors prove that Eq. (8) has a unique solution $x \in W_1^\alpha(0,T;\mathbb{R}^d)$ which is moreover $(1-\alpha)$-Hölder continuous, if $b^i, \sigma^{ij} \in C_b^1$ and the partial derivatives of σ^{ij} are Hölder continuous of order $\lambda > \frac{1}{H} - 1$.

In this section we will show the differentiability of the mapping $g \to x(g)$. For a function φ from $\mathbb{R}^p$ to $\mathbb{R}$, we set $\partial_k\varphi = \frac{\partial\varphi}{\partial x_x}$.

The first step is to establish the existence and uniqueness of a solution for linear equations that are generalizations of (8). The iterated derivatives of the solution of Eq. (8) satisfy such equations.

Proposition 2. *Fix $g \in W_2^{1-\alpha}(0,T;\mathbb{R}^m)$ and consider the following linear equation:*

$$y_t = w_t + \int_0^t B_s y_s \mathrm{d}s + \int_0^t S_s y_s \mathrm{d}g_s, \tag{9}$$

where $w \in C^{1-\alpha}(0,T;\mathbb{R}^d)$, $S \in C^{1-\alpha}(0,T;\mathbb{R}^{d\times d\times m})$ and $B \in C^{1-\alpha}(0,T;\mathbb{R}^{d\times d})$. There exists a unique solution $y \in C^{1-\alpha}(0,T;\mathbb{R}^d)$ of Eq. (9) which satisfies

$$\|y\|_{\alpha,1} \le c_1\, \|w\|_{\alpha,1}\, \exp\left(c_2\, \|g\|_{1-\alpha,2}^{\frac{1}{1-2\alpha}} (\|B\|_\infty + \|S\|_{1-\alpha})\right)\,, \tag{10}$$

where c_1 and c_2 depend only on α and T.

Proof. The existence and uniqueness of a solution can be established following the same lines as in the proof of Theorem 5.1 of [14]. Let us prove the estimate (10). Set $F_t^B = \int_0^t B_s y_s \mathrm{d}s$ and $G_t^S = \int_0^t S_s y_s \mathrm{d}g_s$. Using (4) we have

$$\begin{aligned}\Delta_t^\alpha(G^S) &\le c_{\alpha,T}\, \|g\|_{1-\alpha,2} \int_0^t \left[(t-s)^{-2\alpha} + s^{-\alpha}\right]\Delta_s^\alpha(Sy)\mathrm{d}s \\ &\le c_{\alpha,T}\, \|g\|_{1-\alpha,2} \left(\int_0^t \left[(t-s)^{-2\alpha} + s^{-\alpha}\right]|y_s| \left(\int_0^s \frac{\|S\|_{1-\alpha}(s-r)^{1-\alpha}}{(s-r)^{\alpha+1}}\mathrm{d}r\right)\mathrm{d}s\right.\end{aligned}$$

$$
\begin{aligned}
&+ \ \|S\|_{1-\alpha} \int_0^t \left[(t-s)^{-2\alpha} + s^{-\alpha}\right] \Delta_s^\alpha(y)\mathrm{d}s\Big) \\
&\le c_{\alpha,T} \ \|g\|_{1-\alpha,2} \ \|S\|_{1-\alpha} \ \int_0^t \left[(t-s)^{-2\alpha} + s^{-\alpha}\right] \Delta_s^\alpha(y)\mathrm{d}s,
\end{aligned}
$$

where the constant $c_{\alpha,T}$ may vary from line to line but depends only on α and T. On the other hand, using (6) we get

$$
\Delta_t^\alpha(F^B) \le c_{\alpha,T} \ \|B\|_\infty \int_0^t \frac{|y_s|}{(t-s)^\alpha}\mathrm{d}s.
$$

Then the above inequalities yield that

$$
\Delta_t^\alpha(y) \le \|w\|_{\alpha,\infty} + c_{\alpha,T} \left(\Lambda_\alpha(g) \ \|S\|_{1-\alpha} + \|B\|_\infty \right) \int_0^t \left[(t-s)^{-2\alpha} + s^{-\alpha}\right] h_s\mathrm{d}s.
$$

Applying a Gronwall-type lemma (see Lemma 7.6 in [14]) we derive the estimate (10). ■

The following technical lemma is a basic ingredient in the proof of the Fréchet differentiability of the mapping $x \to x(g)$, where x is the solution of Eq. (8).

Lemma 3. *Let x be the solution of (8). Then the mapping*

$$
F : W_2^{1-\alpha}(0,T;\mathbb{R}^m) \times W_1^\alpha(0,T;\mathbb{R}^d) \to W_1^\alpha(0,T;\mathbb{R}^d)
$$

defined by

$$
(h,x) \mapsto F(h,x) := x - x_0 - \int_0^\cdot b(x_s)\mathrm{d}s - \int_0^\cdot \sigma(x_s)\mathrm{d}(g_s + h_s) \tag{11}
$$

is Fréchet differentiable and we have for any $(h,x) \in W_2^{1-\alpha}(0,T;\mathbb{R}^m) \times W_1^\alpha(0,T;\mathbb{R}^d)$, $k \in W_2^{1-\alpha}(0,T;\mathbb{R}^m)$, $v \in W_1^\alpha(0,T;\mathbb{R}^d)$, and $i = 1,\ldots,d$,

$$
D_1F(h,x)(k)_t^i = -\sum_{j=1}^m \int_0^t \sigma^{ij}(x_s)\mathrm{d}k_s^j, \tag{12}
$$

$$
D_2F(h,x)(v)_t^i = v_t^i - \sum_{k=1}^d \int_0^t \partial_k b(x_s)v_s^k\mathrm{d}s - \sum_{k=1}^d\sum_{j=1}^m \int_0^t \partial_k\sigma^{ij}(x_s)v_s^k\mathrm{d}(g_s^j + h_s^j). \tag{13}
$$

Proof. For (h,x) and $(\tilde{h},\tilde{x})$ in $W_2^{1-\alpha}(0,T;\mathbb{R}^m) \times W_1^\alpha(0,T;\mathbb{R}^d)$ we have

$$
\begin{aligned}
F(h,x)_t - F(\tilde{h},\tilde{x})_t &= x_t - \tilde{x}_t + \int_0^t (b(x_s) - b(\tilde{x}_s))\,\mathrm{d}s \\
&\quad - \int_0^t (\sigma(x_s) - \sigma(\tilde{x}_s))\,(g_s + h_s) - \int_0^t \sigma(\tilde{x}_s)(h_s - \tilde{h}_s).
\end{aligned}
$$

Using Proposition 1 one easily deduces that

$$
\begin{aligned}
\left\|F(h,x) - F(\tilde{h},\tilde{x})\right\|_{\alpha,1} &\le \left(1 + c_{\alpha,T}\|\partial b\|_\infty\right) \ \|x - \tilde{x}\|_{\alpha,\infty} \\
&\quad + c_{\alpha,T} \ \|g + h\|_{1-\alpha,2} \, \|\sigma(x) - \sigma(\tilde{x})\|_{\alpha,\infty} \\
&\quad + c_{\alpha,T}\|\sigma(x)\|_{\alpha,\infty} \ \left\|h - \tilde{h}\right\|_{1-\alpha,2}.
\end{aligned}
$$

Since σ is a Lipschitz function we have $\|\sigma(x)\|_{\alpha,1} \le |\sigma(0)| + \|\partial\sigma\|_\infty \|x\|_{\alpha,1}$. Using the fact that for any x_1, x_2, x_3 and x_4,

$$\begin{aligned}|\sigma(x_1) - \sigma(x_2) - \sigma(x_3) + \sigma(x_4)| \le\ & \|\partial\sigma\|_\infty\, |x_1 - x_2 - x_3 + x_4| \\ & + \|\partial^2\sigma\|_\infty |x_1 - x_3| \left(|x_1 - x_2| + |x_3 - x_4|\right),\end{aligned}$$

it follows that

$$\|\sigma(x) - \sigma(\tilde{x})\|_{\alpha,1} \le \Big(\|\partial\sigma\|_\infty + \|\partial^2\sigma\|_\infty \left(\|x\|_{\alpha,1} + \|\tilde{x}\|_{\alpha,1}\right)\Big)\ \|x - \tilde{x}\|_{\alpha,1}.$$

Consequently, there exists a constant C depending on α, T and the coefficients b and σ such that

$$\begin{aligned}\left\| F(h, x) - F(\tilde{h}, \tilde{x}) \right\|_{\alpha,1} \le\ & \left(1 + C\ \left(\|x\|_{\alpha,1} + \|\tilde{x}\|_{\alpha,1}\right) \|g + h\|_{1-\alpha,2}\right)\ \|x - \tilde{x}\|_{\alpha,1} \\ & + C(1 + \|x\|_{\alpha,1})\ \|h - \tilde{h}\|_{1-\alpha,2},\end{aligned}$$

which implies that F is continuous.

We now prove that it is differentiable with respect to x. Thanks to Proposition 1, it holds that D_2F defined in (13) satisfies

$$\|D_2F(h, x)(v)\|_{\alpha,1} \le c\, \|v\|_{\alpha,1}\ ,$$

and, therefore, it is continuous. Let us now check that for any $h \in W_2^{1-\alpha}(0, T; \mathbb{R}^m)$, D_2F is the Fréchet derivative (with respect to x) of $(h, x) \mapsto F(h, x)$. We have

$$\begin{aligned}& F(h, x + v)_t - F(h, x)_t\ - D_2F(h, x)(v)_t \\ & \quad = \int_0^t (b(x_s) - b(x_s + v_s) + \partial b(x_s)v_s)\, \mathrm{d}s \\ & \qquad + \int_0^t (\sigma(x_s) - \sigma(x_s + v_s) + \partial\sigma(x_s)v_s)\, \mathrm{d}(g_s + h_s).\end{aligned} \tag{14}$$

By the mean value theorem we can write

$$|b(x_s) - b(x_s + v_s) + \partial b(x_s)v_s| \le \|\partial^2 b\|_\infty |v_s|^2$$

and thanks to (7) one easily remarks that

$$\left\| \int_0^\cdot (b(x_s) - b(x_s + v_s) + \partial b(x_s)v_s)\, \mathrm{d}s \right\|_1 \le c_{\alpha,T} \|\partial^2 b\|_\infty\ \|v\|_{\alpha,1}^2.$$

Similar computations for the second term of the right hand side of (14) yield

$$\begin{aligned}& \left\| \int_0^\cdot (\sigma(x_s) - \sigma(x_s + v_s) + \partial\sigma(x_s)v_s)\, \mathrm{d}(g_s + h_s) \right\|_{\alpha,1} \\ & \quad \le c_{\alpha,T} \Big(\|\partial^2\sigma\|_\infty + \|\partial^3\sigma\|_\infty \|x\|_{\alpha,1}\Big)\, \|g + h\|_{1-\alpha,2}\, \|v\|_{\alpha,1}^2.\end{aligned}$$

Thus it follows that

$$\|F(h, x + v) - F(h, x) - D_2F(h, x)(v)\|_{\alpha,1} \le C\ \|g + h\|_{1-\alpha,2}\ \|v\|_{\alpha,1}^2,$$

where C depends on α, T, $\|\partial^2 b\|_\infty$, $\|\partial^2\sigma\|_\infty$, $\|\partial^3\sigma\|_\infty$ and $\|x\|_{\alpha,1}$. Then $(h, x) \mapsto F(h, x)$ is Fréchet differentiable with respect to x and (13) holds. Similar arguments give the differentiability with respect to h and Formula (12). ■

Proposition 4. *Let x be the solution of Eq. (8). Assume $b^i, \sigma^{ij} \in C_b^3$. The mapping $g \to x(g)$ from $W_2^{1-\alpha}(0,T;\mathbb{R}^m)$ into $W_1^{\alpha}(0,T;\mathbb{R}^d)$ is Fréchet differentiable and for any $h \in W_2^{1-\alpha}(0,T;\mathbb{R}^m)$ its derivative in the direction h is given by*

$$D_h x_t^i = \sum_{j=1}^m \int_0^t \Phi_t^{ij}(s)\mathrm{d}h_s^j, \tag{15}$$

where for $i = 1,\dots,d$, $j = 1,\dots,m$, $0 \le s \le t \le T$, $s \mapsto \Phi_t^{ij}(s)$ satisfies

$$\Phi_t^{ij}(s) = \sigma^{ij}(x_s) + \sum_{k=1}^d \int_s^t \partial_k b^i(x_u)\Phi_u^{k,j}(s)\mathrm{d}u + \sum_{k=1}^d \sum_{l=1}^m \int_s^t \partial_k \sigma^{il}(x_u)\Phi_u^{k,j}(s)\mathrm{d}g_u^l, \tag{16}$$

and $\Phi_t^{ij}(s) = 0$ if $s > t$.

Proof. We apply the implicit function theorem to the functional F defined by (11) in Lemma 3. For any (h,x), $F(h,x)$ belongs to $C^{1-\alpha}(0,T;\mathbb{R}^d)$ thanks to Proposition 1. Since x is a solution of (8), one remarks that $F(0,x) = 0$. Thanks to Lemma 3, the mapping F is Fréchet differentiable with first partial derivatives with respect to h given by (12) and the first partial derivative with respect to x is given by (13). We have to check that $D_2F(0,x)$ is a linear homeomorphism from $W_1^{\alpha}(0,T;\mathbb{R}^d)$ to $C^{1-\alpha}(0,T;\mathbb{R}^d)$. By the open map theorem it suffices to show that it is bijective and continuous. We apply Proposition 2 with $t \mapsto B_t = \partial b(x_t)$ and $t \mapsto S_t = \partial\sigma(x_t)$ which are $(1-\alpha)$-Hölder continuous. Thus

$$D_2F(0,x)(v)_t^i = v_t^i - \sum_{k=1}^d \int_0^t \partial_k b^i(x_s)v_s^k \mathrm{d}s - \sum_{k=1}^d \sum_{j=1}^m \int_0^t \partial_k\sigma^{ij}(x_s)v_s^k \mathrm{d}g_s^j$$

is a one-to-one mapping thanks to the existence and uniqueness result for Eq. (8).

Now we fix $w \in C^{1-\alpha}(0,T;\mathbb{R}^d)$. Thanks to Proposition 2, there exists $v \in W_1^{\alpha}(0,T;\mathbb{R}^d)$ such that $w = D_2F(0,x)(v)$; hence $D_2F(0,x)$ is onto and then it is a bijection. We already know that it is continuous. By the implicit function theorem $g \mapsto x(g)$ is continuously Fréchet differentiable and

$$Dx = -D_2F(0,x)^{-1} \circ D_1F(0,x). \tag{17}$$

So for any $k \in W_2^{1-\alpha}(0,T;\mathbb{R}^m)$, $-Dx(k)$ is the unique solution of the differential equation

$$w_t^i = -Dx(k)_t^i + \sum_{k=1}^d \int_0^t \partial_k b(x_s)Dx(k)_s^k \mathrm{d}s + \sum_{k=1}^d \sum_{j=1}^m \int_0^t \partial_k\sigma^{ij}(x_s)Dx(k)_s^k \mathrm{d}g_s^j$$

with $w_t^i = D_1F(0,x)(k)_t = -\sum_{j=1}^m \int_0^t \sigma^{ij}(x_s)\mathrm{d}k_s^j$.

On the other hand, from Eq. (16) we get

$$\sum_{j=1}^m \int_0^t \Phi_t^{ij}(s)\mathrm{d}h_s^j = \sum_{j=1}^m \int_0^t \sigma^{ij}(x_s)\mathrm{d}h_s^j + \sum_{j=1}^m \int_0^t \left(\sum_{k=1}^d \int_s^t \partial_k b^i(x_u)\Phi_u^{k,j}(s)\mathrm{d}u\right)\mathrm{d}s$$
$$+ \sum_{j=1}^m \int_0^t \left(\sum_{k=1}^d \sum_{l=1}^m \int_s^t \partial_k\sigma^{il}(x_u)\Phi_u^{k,j}(s)\mathrm{d}g_u^l\right)\mathrm{d}h_s^j. \tag{18}$$

Using Fubini's theorem we can invert the order of integration in the second integral of the right hand side of (18). The treatment of the third integral is more involved. Thanks to Proposition 9 in the Appendix, $\partial_k \sigma^{il}(x_u)\Phi_u^{k,j}(s)$ is Hölder continuous of order $1-\alpha$ in both variables (u,s). As a consequence, we can apply Fubini's theorem for the Riemann–Stieltjes integrals and we obtain

$$\sum_{j=1}^m \int_0^t \Phi_t^{ij}(s)\mathrm{d}h_s^j = \sum_{j=1}^m \int_0^t \sigma^{ij}(x_s)\mathrm{d}h_s^j + \sum_{k=1}^d \int_0^t \partial_k b^i(x_u)\left(\sum_{j=1}^m \int_0^u \Phi_u^{k,j}(s)\mathrm{d}s\right)\mathrm{d}u \\ + \sum_{k=1}^d \sum_{l=1}^m \int_0^t \partial_k \sigma^{il}(x_u)\left(\sum_{j=1}^m \int_0^u \Phi_u^{k,j}(s)\mathrm{d}h_s^j\right)\mathrm{d}g_u^l.$$

Hence $t \mapsto \sum_{j=1}^m \int_0^t \Phi_t^{ij}(s)\mathrm{d}h_s^j$ is a solution of Eq. (16) and by uniqueness we get the result. ■

If the coefficients b and σ are infinitely differentiable, the mapping $g \to x(g)$ is actually infinitely Fréchet differentiable. The proof of this result uses essentially the same arguments as in the case of first-order derivatives, but the notation is more involved. We state here the result and present the proof in the Appendix.

Proposition 5. *Assume $b^i, \sigma^{ij} \in C_b^\infty$. Then the solution x to Eq. (8) is infinitely continuously Fréchet differentiable. Moreover, for any $(h_1,\ldots,h_n) \in (W_1^{1-\alpha}(0,T;\mathbb{R}^m))^n$, it holds that*

$$D_{h_1,\ldots,h_n} x_t^i = \sum_{i_1,\ldots,i_n=1}^m \int_0^t \cdots \int_0^t \Phi_t^{i,i_1,\ldots,i_n}(r_1,\ldots,r_n)\mathrm{d}h_1^{i_1}(r_1)\mathrm{d}h_2^{i_2}(r_2)\ldots\mathrm{d}h_n^{i_n}(r_n), \quad (19)$$

where the functions $\Phi_t^{i,i_1,\ldots,i_n}(r_1,\ldots,r_n)$ for $t \geq r_1 \vee \cdots \vee r_n$ are defined recursively by

$$\Phi_t^{i,i_1,\ldots,i_n}(r_1,\ldots,r_n) = \sum_{i_0=1}^n A^i_{i_0,i_1,\ldots,i_0-1,i_0+1,\ldots,i_n}(r_{i_0},r_1,\ldots,r_{i_0-1},r_{i_0+1},\ldots,r_n) \\ + \int_{r_1\vee\cdots\vee r_n}^t B^i_{i_1,\ldots,i_n}(r_1,\ldots,r_n;s)\mathrm{d}s \\ + \sum_{l=1}^m \int_{r_1\vee\cdots\vee r_n}^t A^i_{l,i_1,\ldots,i_n}(r_1,\ldots,r_n;s)\mathrm{d}g_s^l, \quad (20)$$

and 0 if $t < r_1 \vee \cdots \vee r_n$. We have defined

$$A^i_{j,i_1,\ldots,i_n}(r_1,\ldots,r_n;s) = \sum_{I_1\cup\cdots\cup I_\nu} \sum_{k_1,\ldots,k_\nu=1}^d \partial_{k_1}\ldots\partial_{k_\nu}\sigma^{ij}(x_s) \\ \times \Phi_s^{k_1,i(I_1)}(r(I_1))\ldots\Phi_s^{k_\nu,i(I_\nu)}(r(I_\nu)),$$

$$B^i_{i_1,\ldots,i_n}(r_1,\ldots,r_n;s) = \sum_{I_1\cup\cdots\cup I_\nu} \sum_{k_1,\ldots,k_\nu=1}^d \partial_{k_1}\ldots\partial_{k_\nu} b^i(x_s) \\ \times \Phi_s^{k_1,i(I_1)}(r(I_1))\ldots\Phi_s^{k_\nu,i(I_\nu)}(r(I_\nu)),$$

where the first sums are extended to the set of all partitions $I_1\cup\cdots\cup I_\nu$ of $\{1,\ldots,n\}$.

3. Stochastic differential equations driven by a fractional Brownian motion

Let $\Omega = C_0([0,T];\mathbb{R}^m)$ be the Banach space of continuous functions, null at time 0, equipped with the supremum norm. Fix $H \in (\frac{1}{2}, 1)$. Let $\mathbb{P}$ be the unique probability measure on Ω such that the canonical process $\{B_t, t \in [0,T]\}$ is an m-dimensional fractional Brownian motion with Hurst parameter H.

We denote by $\mathcal{E}$ the set of step functions on $[0,T]$ with values in $\mathbb{R}^m$. Let $\mathcal{H}$ be the Hilbert space defined as the closure of $\mathcal{E}$ with respect to the scalar product

$$\left\langle \left(\mathbf{1}_{[0,t_1]}, \ldots, \mathbf{1}_{[0,t_m]}\right), \left(\mathbf{1}_{[0,s_1]}, \ldots, \mathbf{1}_{[0,s_m]}\right)\right\rangle_{\mathcal{H}} = \sum_{i=1}^{m} R_H(t_i, s_i).$$

We recall that

$$R_H(t,s) = \int_0^{t\wedge s} K_H(t,r)K_H(s,r)\mathrm{d}r,$$

where $K_H(t,s)$ is the square integrable kernel defined by

$$K_H(t,s) = c_H s^{\frac{1}{2}-H}\int_s^t (u-s)^{H-\frac{3}{2}}u^{H-\frac{1}{2}}\mathrm{d}u$$

for $t > s$, where $c_H = \sqrt{\frac{H(2H-1)}{\beta(2-2H, H-\frac{1}{2})}}$ and β denotes the Beta function. We put $K_H(t,s) = 0$ if $t \le s$.

The mapping $\left(\mathbf{1}_{[0,t_1]}, \ldots, \mathbf{1}_{[0,t_m]}\right) \mapsto \sum_{i=1}^m B_{t_i}^i$ can be extended to an isometry between $\mathcal{H}$ and the Gaussian space $H_1(B)$ spanned by B. We denote this isometry by $\varphi \mapsto B(\varphi)$.

We introduce the operator $K_H^* : \mathcal{E} \to L^2(0,T;\mathbb{R}^m)$ defined by

$$K_H^*\left(\left(\mathbf{1}_{[0,t_1]}, \ldots, \mathbf{1}_{[0,t_m]}\right)\right) = \left(K_H(t_1,.), \ldots, K_H(t_m,.)\right).$$

For any $\varphi, \psi \in \mathcal{E}$, $\langle\varphi,\psi\rangle_{\mathcal{H}} = \langle K_H^*\varphi, K_H^*\psi\rangle_{L^2(0,T;\mathbb{R}^m)} = \mathbb{E}\left(B(\varphi)B(\psi)\right)$ and then K_H^* provides an isometry between the Hilbert space $\mathcal{H}$ and a closed subspace of $L^2(0,T;\mathbb{R}^m)$.

We denote as $\mathcal{K}_H : L^2(0,T;\mathbb{R}^m) \to \mathcal{H}_H := \mathcal{K}_H\left(L^2(0,T;\mathbb{R}^m)\right)$ the operator defined by

$$(\mathcal{K}_H h)(t) := \int_0^t K_H(t,s)h(s)\mathrm{d}s\ .$$

The space $\mathcal{H}_H$ is the fractional version of the Cameron–Martin space. In the case of a classical Brownian motion, $K_H(t,s) = \mathbf{1}_{[0,t]}(s)$, $\mathcal{K}_H^*$ is the identity map on $L^2(0,T;\mathbb{R}^m)$, and $\mathcal{H}_H$ is the space of absolutely continuous functions, vanishing at zero, with a square integrable derivative.

We finally denote by $\mathcal{R}_H = \mathcal{K}_H \circ \mathcal{K}_H^* : \mathcal{H} \to \mathcal{H}_H$ the operator

$$\mathcal{R}_H\varphi = \int_0^{\cdot} K_H(\cdot,s)\left(\mathcal{K}_H^*\varphi\right)(s)\mathrm{d}s.$$

We remark that for any $\varphi \in \mathcal{H}$, $\mathcal{R}_H\varphi$ is Hölder continuous of order H. Indeed,

$$\left(\mathcal{R}_H\varphi\right)^i(t) = \int_0^T \left(\mathcal{K}_H^*\mathbf{1}_{[0,t]}\right)^i(s)\left(\mathcal{K}_H^*\varphi\right)^i(s)\mathrm{d}s = \mathbb{E}\left(B_t^i B^i(\varphi)\right),$$

and consequently

$$\left|\left(\mathcal{R}_H\varphi\right)^i(t) - \left(\mathcal{R}_H\varphi\right)^i(s)\right| \le \left(\mathbb{E}\left(|B_t^i - B_s^i|^2\right)\right)^{1/2}\|\varphi\|_{\mathcal{H}} \le \|\varphi\|_{\mathcal{H}}\,|t-s|^H.$$

On the other hand, for any $\varphi \in \mathcal{H}$, $\mathcal{R}_H\varphi$ is absolutely continuous and

$$(\mathcal{R}_H\varphi)(t) = \int_0^t \left(\int_0^u \frac{\partial K_H}{\partial u}(u,s) \left(\mathcal{K}_H^*\varphi\right)(s)\mathrm{d}s \right) \mathrm{d}u. \tag{21}$$

This equation follows from Fubini's theorem and the fact that $\frac{\partial K_H}{\partial u}(u,s) = c_H \left(\frac{u}{s}\right)^{H-\frac{1}{2}} (u-s)^{H-\frac{3}{2}}$. Notice also that $\mathcal{R}_H \mathbf{1}_{[0,t]} = R_H(t,\cdot)$, and, as a consequence, $\mathcal{H}_H$ is the reproducing kernel Hilbert space associated with the Gaussian process B.

The injection $\mathcal{R}_H : \mathcal{H} \to \Omega$ embeds $\mathcal{H}$ densely into Ω and for any $\varphi \in \Omega^* \subset \mathcal{H}$ we have

$$\mathbb{E}\left(\mathrm{e}^{\mathrm{i}\langle B,\varphi\rangle}\right) = \exp\left(-\frac{1}{2}\|\varphi\|_{\mathcal{H}}^2\right).$$

As a consequence, $(\Omega, \mathcal{H}, \mathbb{P})$ is an abstract Wiener space in the sense of Gross. Notice that the choices of the Hilbert space and its embedding into Ω are not unique and in [5] the authors have made another (but equivalent) choice for the underlying Hilbert space.

Let $\{X_t, t \in [0,T]\}$ be the solution of the stochastic differential equation (2), and assume that the coefficients are infinitely differentiable, bounded together with all their derivatives. We fix $1 - H < \alpha < \frac{1}{2}$. The trajectories of the fractional Brownian motion belong almost surely to $C^{1-\alpha+\varepsilon}(0,T;\mathbb{R}^m) \subset W_2^{1-\alpha}(0,T;\mathbb{R}^m)$ if $\varepsilon < H + \alpha - 1$. Therefore, by Proposition 5, the mapping $\omega \mapsto X(\omega)$ is infinitely Fréchet differentiable from $W_2^{1-\alpha}(0,T;\mathbb{R}^m)$ into $W_1^{\alpha}(0,T;\mathbb{R}^d)$.

On the other hand, we have seen that $\mathcal{H}_H \subset C^H(0,T;\mathbb{R}^m) \subset W_2^{1-\alpha}(0,T;\mathbb{R}^m)$. As a consequence, the following iterated derivative exists:

$$D_{\mathcal{R}_H\varphi_1,\ldots,\mathcal{R}_H\varphi_1} X_t^i = \frac{\mathrm{d}}{\mathrm{d}\varepsilon_1}\cdots\frac{\mathrm{d}}{\mathrm{d}\varepsilon_n} X_t^i(\omega + \varepsilon_1\mathcal{R}_H\varphi_1 + \cdots + \varepsilon_n\mathcal{R}_H\varphi_n)|_{\varepsilon_1=\cdots=\varepsilon_n=0},$$

for all $\varphi_i \in \mathcal{H}$. In this way we have proved the following result.

Theorem 6. *Let $H > 1/2$ and assume that $b^i, \sigma^{ij} \in C_b^3$. Then the stochastic process X solution of the stochastic differential equation (2) is almost surely differentiable in the directions of the Cameron–Martin space. If $b^i, \sigma^{ij} \in C_b^\infty$, then X is almost surely infinitely differentiable in the directions of the Cameron–Martin space.*

The iterated derivative $D_{\mathcal{R}_H\varphi_1,\ldots,\mathcal{R}_H\varphi_1} X_t^i$ coincides with $\left\langle D^n X_t^i, \varphi_1 \otimes \cdots \otimes \varphi_n \right\rangle_{\mathcal{H}^\otimes}$, where D^n is the iterated derivative in the Malliavin calculus sense. In fact, if F is a smooth cylindrical random variable of the form

$$F = f(B(\varphi_1), \ldots, B(\varphi_m))$$

with $f \in C_b^\infty(\mathbb{R}^m)$, $\varphi_i \in \mathcal{H}$, then the Malliavin derivative DF is the $\mathcal{H}$-valued random variable defined by

$$\begin{aligned}\langle DF, h\rangle_{\mathcal{H}} &= \sum_{i=1}^m \partial_i f(B(\varphi_1), \ldots, B(\varphi_m))\langle \varphi_i, h\rangle_{\mathcal{H}} \\ &= \frac{\mathrm{d}}{\mathrm{d}\varepsilon} f\left(B(\varphi_1) + \varepsilon\langle\varphi_1, h\rangle_{\mathcal{H}}, \ldots, B(\varphi_m) + \varepsilon\langle\varphi_m, h\rangle_{\mathcal{H}}\right)|_{\varepsilon=0},\end{aligned}$$

and one can easily see that

$$B(\varphi_1)(\omega + \varepsilon\mathcal{R}_H h) = B(\varphi_1)(\omega) + \varepsilon\langle\varphi_1, h\rangle_{\mathcal{H}}.$$

We recall here that $\mathbb{D}^{k,p}$ is the closure of the space of smooth and cylindrical random variables with respect to the norm

$$\|F\|_{k,p} = \left[E(|F|^p) + \sum_{j=1}^{k} E(\|D^j F\|^p_{\mathcal{H}^{\otimes j}}) \right]^{\frac{1}{p}},$$

and $\mathbb{D}^{k,p}_{\text{loc}}$ is the set of random variables F such that there exist a sequence $\{(\Omega_n, F_n), n \geq 1\}$ such that $\Omega_n \uparrow \Omega$ a.s., $F_n \in \mathbb{D}^{k,p}$ and $F = F_n$ a.s. on Ω_n.

By the results of Kusuoka (see [7], Theorem 5.2 or [12], Proposition 4.1.3), Theorem 6 implies that X^i_t belongs to the space $\mathbb{D}^{k,p}_{\text{loc}}$ for all $p > 1$ and any integer k.

Now we give the equations satisfied by the derivatives of the process X.

Proposition 7. *If we define $(i_1, \ldots, i_n) \in \{1, \ldots, m\}^n$, a multi-index, the n-th derivative in the sense of Malliavin calculus satisfies the following linear equation a.s.:*

$$\begin{aligned} D^{i_1,\ldots,i_n}_{r_1,\ldots,r_n} X^i_t = {} & \sum_{i_0=1}^{n} \alpha^i_{i_0,i_1,\ldots,i_0-1,i_0+1,\ldots,i_n}(r_{i_0}, r_1, \ldots, r_{i_0-1}, r_{i_0+1}, \ldots, r_n) \\ & + \int_{r_1\vee\cdots\vee r_n}^{t} \beta^i_{i_1,\ldots,i_n}(r_1, \ldots, r_n; s)\mathrm{d}s \\ & + \sum_{l=1}^{m} \int_{r_1\vee\cdots\vee r_n}^{t} \alpha^i_{l,i_1,\ldots,i_n}(r_1, \ldots, r_n; s)\mathrm{d}B^l_s \end{aligned} \tag{22}$$

if $t \geq r_1 \vee \cdots \vee r_n$, and $D^{i_1,\ldots,i_n}_{r_1,\ldots,r_n} X^i_t = 0$ otherwise. In the above equation, we have defined

$$\alpha^i_{j,i_1,\ldots,i_n}(r_1, \ldots, r_n; s) = \sum_{I_1\cup\cdots\cup I_\nu} \sum_{k_1,\ldots,k_\nu=1}^{d} \partial_{k_1} \ldots \partial_{k_\nu} \sigma^{ij}(X_s)\, D^{i(I_1)}_{r(I_1)} X^{k_1}_s \cdots D^{i(I_\nu)}_{r(I_\nu)} X^{k_\nu}_s,$$

$$\beta^i_{i_1,\ldots,i_n}(r_1, \ldots, r_n; s) = \sum_{I_1\cup\cdots\cup I_\nu} \sum_{k_1,\ldots,k_\nu=1}^{d} \partial_{k_1} \ldots \partial_{k_\nu} b^{i}(X_s)\, D^{i(I_1)}_{r(I_1)} X^{k_1}_s \cdots D^{i(I_\nu)}_{r(I_\nu)} X^{k_\nu}_s,$$

where the first sums are extended to the set of all partitions $I_1 \cup \cdots \cup I_\nu$ of $\{1, \ldots, n\}$ and for any subset $K = \{i_1, \ldots, i_\eta\}$ of $\{1, \ldots, n\}$, we put $D^{i(K)}_{r(K)}$ the derivative operator $D^{i_1,\ldots,i_\eta}_{r_{i_1},\ldots,r_{i_\eta}}$.

For the first-order derivative, Eq. (22) reads as follows: for $i = 1, \ldots, d$, $j = 1, \ldots, m$,

$$D^j_s X^i_t = \sigma^{ij}(X_s) + \sum_{k=1}^{d} \int_s^t \partial_k b^i(X_u) D^j_s X^k_u \mathrm{d}u + \sum_{k=1}^{d}\sum_{l=1}^{m} \int_s^t \partial_k \sigma^{il}(X_u) D^j_s X^k_u \mathrm{d}B^l_u,$$

if $s \leq t$ and 0 if $s > t$.

Proof. We use the representation result on the deterministic equation given by (19) in Proposition 5. For any $h = (h_1, \ldots, h_n)$ with $h_i \in \mathcal{H}$, we have

$$\begin{aligned} D_{\mathcal{R}_H h_1,\ldots,\mathcal{R}_H h_n} X^i_t = {} & \sum_{i_1,\ldots,i_n=1}^{m} \int_0^t \cdots \int_0^t \Phi^{i,i_1,\ldots,i_n}_t(r_1, \ldots, r_n) \\ & \times \mathrm{d}(\mathcal{R}_H h_1)^{i_1}(r_1) \cdots \mathrm{d}(\mathcal{R}_H h_n)^{i_n}(r_n). \end{aligned} \tag{23}$$

We denote as $\mathcal{K}_H^{*\otimes n}$ the map from $\mathcal{H}^{\otimes n}$ into $\left(L^2(0,T;\mathbb{R}^m)\right)^{\otimes n}$ defined for $\varphi \in \mathcal{H}^{\otimes n}$ by

$$\left(\mathcal{K}_H^{*\otimes n}\varphi\right)(s_1,\dots,s_n) = \int_{s_1}^T \cdots \int_{s_n}^T \varphi(r_1,\dots,r_n) \\ \times \frac{\partial K_H}{\partial r_1}(r_1,s_1)\cdots\frac{\partial K_H}{\partial r_n}(r_n,s_n)\mathrm{d}r_1\cdots\mathrm{d}r_n.$$

It holds that

$$\langle\varphi,\psi\rangle_{\mathcal{H}^{\otimes n}} = \sum_{\xi\in\{1,\dots,m\}^n}\int_{[0,T]^n}\left(\mathcal{K}_H^{*\otimes n}\varphi\right)^{\xi}(s_1,\dots,s_n) \\ \times \left(\mathcal{K}_H^{*\otimes n}\psi\right)^{\xi}(s_1,\dots,s_n)\mathrm{d}s_1\dots\mathrm{d}s_n\ .$$

Thanks to Step 3 in the proof Proposition 5, for any $1\le k\le n$,

$$s_k \mapsto \int_{[0,t]^{k-1}} \Phi_t^{i,i_1,\dots,i_n}(s_1,\dots,s_{k-1},s_k,s_{k+1},\dots,s_n)\mathrm{d}h_1^{i_1}(s_1)\cdots\mathrm{d}h_{k-1}^{i_{k-1}}(s_{k-1})$$

belongs to $C^{1-\alpha}(0,T)$ and applying Eq. (21) n times yields almost surely

$$\begin{aligned} &D_{\mathcal{R}_H h_1,\dots,\mathcal{R}_H h_n}X_t^i \\ &= \sum_{i_1,\dots,i_n=1}^m \int_0^t\cdots\int_0^t \Phi_t^{i,i_1,\dots,i_n}(r_1,\dots,r_n)\left(\int_0^{r_1}\frac{\partial K_H}{\partial r_1}(r_1,u)\left(\mathcal{K}_H^* h_1\right)^{i_1}(u)\mathrm{d}u\right) \\ &\quad\times\cdots\times\left(\int_0^{r_n}\frac{\partial K_H}{\partial r_n}(r_n,u)\left(\mathcal{K}_H^* h_1\right)^{i_n}(u)\mathrm{d}u\right)\mathrm{d}r_1\cdots\mathrm{d}r_n \\ &= \sum_{i_1,\dots,i_n=1}^m \int_0^T\cdots\int_0^T\left(\mathcal{K}_H^{*\otimes n}\Phi_t^i\right)^{i_1,\dots,i_n}(s_1,\dots,s_n)\prod_{l=1}^n\left(\mathcal{K}_H^* h_l\right)^{i_l}(s_i)\mathrm{d}s_1\dots\mathrm{d}s_n \\ &= \left\langle\Phi_t^i, h_1\otimes\cdots\otimes h_n\right\rangle_{\otimes n}, \end{aligned}$$

and the result is proved. ■

4. Absolute continuity of the law of the solution

The fact that for $H>\frac{1}{2}$, the solution of Eq. (8) belongs to the localized domain of the Malliavin derivative operator D will imply the absolute continuity of the law of X_t for all $T>0$ under suitable nondegeneracy conditions.

Theorem 8. *Let $H > 1/2$ and assume that $b^i,\sigma^{ij}\in C_b^3$. Suppose that the following nondegeneracy condition on the coefficient σ holds:*

(H) *The vector space spanned by $\left\{\left(\sigma^{1j}(x_0),\dots,\sigma^{dj}(x_0)\right), 1\le j\le m\right\}$ is $\mathbb{R}^d$.*

Then for any $t>0$, the law of the random vector X_t is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^d$.

Proof. We already know by Theorem 6 that X_t^i belongs to $\mathbb{D}_{\mathrm{loc}}^{1,2}$ for all $t\in[0,T]$ and for $i=1,\dots,d$. Then, thanks to [12], Theorem 2.1.2 due to Bouleau and Hirsch, it suffices to show that the Malliavin covariance matrix of X_t defined by

$$Q_t^{ij} = \left\langle DX_t^i, DX_t^j\right\rangle_{\mathcal{H}}$$

is invertible almost surely. We first deduce another expression for the matrix Q_t. We stress the fact that $\mathcal{K}_H^*$ is an isometry between $\mathcal{H}$ and a closed subspace of $L^2(0,T;\mathbb{R}^m)$. Let $\{e_n, n \geq 1\}$ be a complete orthonormal system in this closed subspace. The elements $f_n = \left(\mathcal{K}_H^*\right)^{-1}(e_n)$ form a complete orthonormal system of $\mathcal{H}$. Then it holds almost surely that

$$DX_t^i = \sum_{n\geq 1} \left\langle DX_t^i, f_n \right\rangle_{\mathcal{H}} f_n,$$

and consequently

$$Q_t^{ij} = \sum_{n\geq 1} \left\langle DX_t^i, f_n \right\rangle_{\mathcal{H}} \left\langle DX_t^j, f_n \right\rangle_{\mathcal{H}}.$$

Suppose now that the Malliavin covariance matrix is not almost surely invertible, that is $\mathbb{P}(\det Q_t = 0) > 0$. Then there exists a vector $v \in \mathbb{R}^d$, $v \neq 0$, such that $v^T Q_t v = 0$. Our aim is to prove that condition (H) cannot be satisfied. One may write

$$v^T Q_t v = \sum_{n\geq 1} \left| \left\langle \langle DX_t, f_n \rangle_{\mathcal{H}}, v \right\rangle_{\mathbb{R}^d} \right|^2.$$

From (15) and (23) it follows that

$$\left\langle DX_t^i, f_n \right\rangle_{\mathcal{H}} = D_{\mathcal{R}_H f_n} X_t^i,$$

and thanks to the representation (17), the directional derivative $D_{\mathcal{R}_H f_n} X_t^i$ satisfies

$$D_{\mathcal{R}_H f_n} X_t^i = \left(D_2 F(0,X)^{-1} \circ D_1 F(0,X) \right) (\mathcal{R}_H f_n)_t^i.$$

It follows that

$$0 = \left\langle \left(D_2 F(0,X)^{-1} \circ D_1 F(0,X) \right) (\mathcal{R}_H f_n)_t, v \right\rangle_{\mathbb{R}^d}.$$

Since $D_2 F(0,X)^{-1}$ is a linear homeomorphism,

$$0 = \langle D_1 F(0,X)(\mathcal{R}_H f_n)_t, v \rangle_{\mathbb{R}^d} = \sum_{i=1}^{d} \left(\sum_{j=1}^{m} \int_0^t \sigma^{ij}(X_s) \mathrm{d}(\mathcal{R}_H f_n)_s^j \right) v^i$$

$$= \left\langle \sum_{i=1}^{d} v^i \, \sigma^i(X) \mathbf{1}_{[0,t]}, f_n \right\rangle_{\mathcal{H}}$$

holds true for any $n \geq 1$ (where σ^i denotes the ith row of the matrix σ). Then

$$0 = \left\| \sum_{i=1}^{d} v^i \, \sigma^i(X) \mathbf{1}_{[0,t]} \right\|_{\mathcal{H}}$$

and this yields that for all $j = 1, \ldots, m$ and $s \in [0,t]$

$$\sum_{i=1}^{d} v^i \, \sigma^{ij}(X_s) = 0.$$

Taking $s = 0$ we get $\sum_{i=1}^{d} v^i \, \sigma^{ij}(x_0) = 0$ for all $j = 1, \ldots, m$ and this contradicts (H). Then the law of the solution of the stochastic differential equation (2) at any time $t > 0$ is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^d$. ∎

Acknowledgement

This work was carried out during a stay of Bruno Saussereau at the IMUB (Institut de Matemàtica de la Universitat de Barcelona). He would like to thank the IMUB for its hospitality and support.

We would like to thank an anonymous referee for helpful remarks.

The first author was partially supported by the NSF Grant DMS 0604207.

Appendix

The next proposition provides the joint continuity property of the solution of equations similar to Eq. (16) satisfied by the kernel of the derivative.

Proposition 9. *Fix $\gamma, B, S \in C^{1-\alpha}(0,T)$ and $g \in W_2^{1-\alpha}(0,T)$ and consider the equation*

$$\rho_t(s) = \gamma(s) + \int_s^t B_u \rho_u(s) \mathrm{d}u + \int_s^t S_u \rho_u(s) \mathrm{d}g_u \tag{24}$$

if $s \le t$ and $\rho_t(s) = 0$ if $s > t$. Then the solution is a Hölder continuous function of order $1-\alpha$ in both variables.

Proof. First notice that $\|\rho_\cdot(s)\|_{\alpha,1}$ is uniformly bounded in s, by the estimate (10). Hence, the function $\rho_t(s)$ is Hölder continuous in t, uniformly in s by (5) and (7). On the other hand, for $s' \le s \le t$ we have

$$\rho_t(s) - \rho_t(s') = w(s,s') + \int_s^t B_u \left(\rho_u(s) - \rho_u(s')\right) \mathrm{d}u + \int_s^t S_u \left(\rho_u(s) - \rho_u(s')\right) \mathrm{d}g_u, \tag{25}$$

where

$$w(s,s') = \gamma(s) - \gamma(s') + \int_{s'}^s B_u \rho_u(s) \mathrm{d}u + \int_{s'}^s S_u \rho_u(s) \mathrm{d}g_u. \tag{26}$$

Proposition 2 yields the estimate

$$\sup_{s\in[0,T]} \|\rho_\cdot(s)\|_{\alpha,1} \le c_1 \|\gamma\|_{\alpha,1} \exp\left(c_2 \|g\|_{1-\alpha,2}^{\frac{1}{1-2\alpha}} (\|B\|_\infty + \|S\|_{1-\alpha})\right). \tag{27}$$

Substituting (27) into (26) yields

$$\begin{aligned} |w(s,s')| &\le \|\gamma\|_{1-\alpha}(s-s')^{1-\alpha} + \|B\|_\infty \left(\sup_s \|\rho_\cdot(s)\|_{\alpha,1}\right)(s-s') \\ &\quad + c\|S\|_{\alpha,1}\, \|g\|_{1-\alpha,2} \left(\sup_s \|\rho_\cdot(s)\|_{\alpha,1}\right)(s-s')^{1-\alpha} \\ &\le c_1\, (s-s')^{1-\alpha}. \end{aligned} \tag{28}$$

Then Proposition 2 applied to Eq. (25) and the estimate (28) imply that

$$\|\rho_\cdot(s) - \rho_\cdot(s')\|_{\alpha,1} \le c_1\ (s-s')^{1-\alpha}\ \exp\left(c_2 \|g\|_{1-\alpha,2}^{\frac{1}{1-2\alpha}} (\ \|B\|_\infty + \|S\|_{1-\alpha})\right).$$

Therefore, $\rho_t(s)$ is Hölder continuous in the variable s, uniformly in t. This completes the proof of the proposition. ■

For the proof of Proposition 5 we need the following technical lemma.

Lemma 10. *Suppose that we are given a mapping $g \mapsto v^g$ from $W_2^{1-\alpha}(0,T;\mathbb{R}^m)$ to $W_1^\alpha(0,T;\mathbb{R}^M)$ which is continuously Fréchet differentiable. Consider five bounded differentiable functions $a_0,\dots,a_4$ from $\mathbb{R}^d$ to $\mathbb{R}^{d\times m\times M}$, $\mathbb{R}^{d\times M}$, $\mathbb{R}^{d\times d}$, $\mathbb{R}^{d\times m\times M}$ and $\mathbb{R}^{d\times m\times d}$, respectively. We moreover assume that these functions have bounded derivatives up to order 2. Let $y \in W_1^\alpha(0,T;\mathbb{R}^d)$ be the solution of the following equation:*

$$y_t = \int_0^t a_0(x_r^g)v_r^g \,\mathrm{d}k_r + \int_0^t \left\{a_1(x_r^g)v_r^g + a_2(x_r^g)y_r\right\}\mathrm{d}r + \int_0^t \left\{a_3(x_r^g)v_r^g + a_4(x_r^g)y_r\right\}\mathrm{d}g_r, \quad (29)$$

where $k \in W_2^{1-\alpha}(0,T;\mathbb{R}^m)$ and x^g is the unique solution of (8) which is already continuously Fréchet differentiable.

Then $g \mapsto y$ is continuously Fréchet differentiable and the directional derivative in the direction $h \in W_2^{1-\alpha}(0,T;\mathbb{R}^m)$ is the unique solution of

$$\begin{aligned} D_h y_t = &\int_0^t \{\partial a_0(x_r)D_h x_r v_r + a_0(x_r)D_h v_r\}\,\mathrm{d}k_r + \int_0^t \{a_3(x_r)v_r + a_4(x_r)y_r\}\,\mathrm{d}h_r \\ &+ \int_0^t \{\partial a_1(x_r)D_h x_r v_r + a_1(x_r)D_h v_r + \partial a_2(x_r)D_h x_r y_r + a_2(x_r)D_h y_r\}\,\mathrm{d}r \\ &+ \int_0^t \{\partial a_3(x_r)D_h x_r v_r + a_3(x_r)D_h v_r + \partial a_4(x_r)D_h x_r y_r + a_4(x_r)D_h y_r\}\,\mathrm{d}g_r. \quad (30) \end{aligned}$$

Proof. We introduce the map

$$\begin{aligned} &W_2^{1-\alpha}(0,T;\mathbb{R}^m)\times W_1^\alpha(0,T;\mathbb{R}^d) \to C^{1-\alpha}(0,T;\mathbb{R}^d) \subset W_1^\alpha(0,T;\mathbb{R}^d) \\ &(h,y) \mapsto F(h,y)(t) := y_t - \int_0^t a_0(x_r^{g+h})v_r^{g+h}\mathrm{d}k_r \\ &\qquad - \int_0^t \left\{a_1(x_r^{g+h})v_r^{g+h} + a_2(x_r^{g+h})y_r\right\}\mathrm{d}r \\ &\qquad - \int_0^t \left\{a_3(x_r^{g+h})v_r^{g+h} + a_4(x_r^{g+h})y_r\right\} d(g_r+h_r). \end{aligned}$$

One has $F(0,y) = 0$ since y is the solution of (29). As in Lemma 3 we can show that F is Fréchet differentiable and

$$\begin{aligned} D_1F(0,y)_t = &-\int_0^t \{\partial a_0(x_r)Dx(h)_r v_r + a_0(x_r)Dv(h)_r\}\,\mathrm{d}k_r \\ &-\int_0^t \{a_3(x_r)v_r + a_4(x_r)y_r\}\,\mathrm{d}h_r \\ &-\int_0^t \{\partial a_1(x_r)Dx(h)_r v_r + a_1(x_r)Dv(h)_r + \partial a_2(x_r)Dx(h)_r\ y_r\}\,\mathrm{d}r \\ &-\int_0^t \{\partial a_3(x_r)Dx(h)_r v_r + a_3(x_r)Dv(h)_r + \partial a_4(x_r)Dx(h)_r\ y_r\}\,\mathrm{d}g_r \\ D_2F(0,y)(z)_t = &\ z_t - \int_0^t a_4(x_r)z_r\mathrm{d}g_r - \int_0^t a_2(x_r)z_r\mathrm{d}r, \end{aligned}$$

for any $z \in W_1^\alpha(0,T;\mathbb{R}^d)$. Then, using Proposition 2 and the same arguments as in the proof of Proposition 4 we conclude that $g \mapsto y$ is continuously Fréchet differentiable and it has a directional derivative in the direction h satisfying (30). ■

Proof of Proposition 5. The proof of Proposition 5 is divided into several steps. We begin by proving that x is infinitely Fréchet differentiable. Then we show that Eq. (20) has a unique solution and derive some of its properties. Finally we prove that (19) holds.

Step 1. We begin by proving by induction that x is infinitely Fréchet continuously differentiable. We introduce some notation in order to write the equations satisfied by the higher order directional derivatives.

Let $n \geq 1$ and for $i = 1, \dots, n$, $h_i = (h_i^1, \dots, h_i^m) \in W_2^{1-\alpha}(0, T; \mathbb{R}^m)$. For any subset $K = \{\varepsilon_1, \dots, \varepsilon_\eta\}$ of $\{1, \dots, n\}$, we denote by $D_{j(K)}$ the iterated directional derivative

$$D_{j(K)}x = D_{h_{\varepsilon_1},\dots,h_{\varepsilon_\eta}}x = D^\eta x(h_{\varepsilon_1}, \dots, h_{\varepsilon_\eta}),$$

where D^η denotes the iterated Fréchet derivative of order η. Define for $i = 1, \dots, d$ and $j = 1, \dots, m$

$$\alpha^{ij}(h_1, \dots, h_n; s) = \sum_{I_1 \cup \dots \cup I_\nu} \sum_{k_1,\dots,k_\nu=1}^{d} \partial_{k_1} \dots \partial_{k_\nu} \sigma^{ij}(x_s)\, D_{j(I_1)}x_s^{k_1} \cdots D_{j(I_\nu)}x_s^{k_\nu},$$

$$\beta^{i}(h_1, \dots, h_n; s) = \sum_{I_1 \cup \dots \cup I_\nu} \sum_{k_1,\dots,k_\nu=1}^{d} \partial_{k_1} \dots \partial_{k_\nu} b^{i}(x_s)\, D_{j(I_1)}x_s^{k_1} \cdots D_{j(I_\nu)}x_s^{k_\nu},$$

where the first sums are extended to the set of all partitions $I_1 \cup \dots \cup I_\nu$ of $\{1, \dots, n\}$. The n-th iterated derivative satisfies the following linear equation:

$$\begin{aligned} D_{h_1,\dots,h_n}x_t^i &= \sum_{j_0=1}^{n} \sum_{j=1}^{m} \int_0^t \alpha^{ij}(h_1, \dots, h_{j_0-1}, h_{j_0+1}, \dots, h_n; s)\mathrm{d}h_{j_0}^j(s) \\ &\quad + \int_0^t \beta^i(h_1, \dots, h_n; s)\mathrm{d}s + \sum_{j=1}^{m} \int_0^t \alpha^{ij}(h_1, \dots, h_n; s)\mathrm{d}g_s^j. \end{aligned} \tag{31}$$

for $i = 1, \dots, n$. Now we prove by induction that x is infinitely Fréchet differentiable and (31) holds. The result is true for $n = 1$ thanks to Proposition 4. Suppose that these properties hold up to the index n. Observe that $\alpha^{ij}(h_1, \dots, h_n; s)$ is equal to the term corresponding to $\nu = 1$, namely

$$\sum_{k=1}^{d} \partial_k \sigma^{ij}(x_s) D_{h_1,\dots,h_n} x_s^k,$$

plus a polynomial function on the derivatives $\partial_{k_1} \dots \partial_{k_\nu} \sigma^{ij}(x_s)$ with $\nu \geq 2$, and the functions $D_{j(I)}x_s$, with $\mathrm{card}(I) \leq n - 1$. Therefore, we can apply Lemma 10 with $y = D_{h_1,\dots,h_n}x$, v the vector function whose entries are the products $D_{j(I_1)}x_s^{k_1} \dots D_{j(I_\nu)}x_s^{k_\nu}$ for all the partitions $I_1 \cup \dots \cup I_\nu$ with $\nu \geq 2$ and with appropriate functions a_i, $i = 0, \dots, 4$. This lemma yields that $g \mapsto D_{h_1,\dots,h_n}x$ is continuously Fréchet differentiable and the directional derivative of order $n + 1$ is solution of (31) at the rank $n + 1$.

Let us now prove by induction that the map $(h_1, \dots, h_n) \mapsto D_{(h_1,\dots,h_n)}x$ is multi-linear and continuous. By Proposition 4 this is true for $n = 1$. Suppose it holds up to $n - 1$, that is, for any subset $\{\varepsilon_1, \dots, \varepsilon_{n_0}\}$ of $\{1, \dots, n\}$ with $n_0 < n$, the maps $(h_{\varepsilon_1}, \dots, h_{\varepsilon_{n_0}}) \mapsto D_{h_{\varepsilon_1},\dots,h_{\varepsilon_{n_0}}}x$ are multi-linear and continuous. We define

$$w^i(h_1,\ldots,h_n;g)_t = \sum_{j_0=1}^{n}\sum_{j=1}^{m}\int_0^t \alpha^{ij}(h_1,\ldots,h_{j_0-1},h_{j_0+1},\ldots,h_n;s)\mathrm{d}h_{j_0}^j(s)$$
$$+\int_0^t\left(\sum_{\substack{I_1\cup\cdots\cup I_\nu\\ \nu>1}}\sum_{k_1,\ldots,k_\nu=1}^{d}\partial_{k_1}\ldots\partial_{k_\nu}b^i(x_s)\, D_{j(I_1)}x_s^{k_1}\ldots D_{j(I_\nu)}x_s^{k_\nu}\right)\mathrm{d}s$$
$$+\sum_{j=1}^{m}\int_0^t\left(\sum_{\substack{I_1\cup\cdots\cup I_\nu\\ \nu>1}}\sum_{k_1,\ldots,k_\nu=1}^{d}\partial_{k_1}\ldots\partial_{k_\nu}\sigma^i(x_s)D_{j(I_1)}x_s^{k_1}\ldots D_{j(I_\nu)}x_s^{k_\nu}\right)\mathrm{d}g_s^j,$$

and using (10) we have the following estimate:

$$\|D_{h_1,\ldots,h_n}x\|_{\alpha,1}\le C\,\|w\|_{\alpha,1},$$

where the constant C does not depend on $(h_1,\ldots,h_n)$. Using the induction hypothesis, we easily deduce that for any $i_0\in\{1,\ldots,n\}$

$$\|w\|_{\alpha,1}\le C_{h_1,\ldots,h_{i_0}-1,h_{i_0}+1,\ldots,h_n}\,\|h_{i_0}\|_{1-\alpha,2}.$$

So the map $h_{i_0}\mapsto D_{h_1,\ldots,h_{i_0},\ldots,h_n}x_\cdot$ is continuous for any $i_0\in\{1,\ldots,n\}$. This map is clearly linear thanks to the induction hypothesis and the existence and uniqueness result of Proposition 2.

It only remains to prove that $g\mapsto D^nx(g)$, from $W_2^{1-\alpha}(0,T;\mathbb{R}^d)$ to the space of multi-linear continuous applications on $W_2^{1-\alpha}(0,T;\mathbb{R}^m)$, is continuous. We proceed again by induction. We write the difference between the two equations satisfied respectively by $D_{h_1,\ldots,h_n}x(g)$ and $D_{h_1,\ldots,h_n}x(\tilde{g})$ (with $b=0$ for ease of reading)

$$\begin{aligned}&D_{h_1,\ldots,h_n}x(g)_t - D_{h_1,\ldots,h_n}x(\tilde{g})_t\\ &\quad = w(h_1,\ldots,h_n;g)_t - w(h_1,\ldots,h_n;\tilde{g})_t\\ &\qquad+\sum_{k=1}^{d}\sum_{j=1}^{m}\int_0^t\left(\partial_k\sigma^{ij}(x(g)_s)-\partial_k\sigma^{ij}(x(\tilde{g})_s)\right)D_{h_1,\ldots,h_n}x(\tilde{g})_s\mathrm{d}g_s\\ &\qquad+\sum_{k=1}^{d}\sum_{j=1}^{m}\int_0^t\partial_k\sigma^{ij}(x(\tilde{g})_s)D_{h_1,\ldots,h_n}x(\tilde{g})_s\mathrm{d}(g_s-\tilde{g}_s)\\ &\qquad+\sum_{k=1}^{d}\sum_{j=1}^{m}\int_0^t\partial_k\sigma^{ij}(x(g)_s)\left(D_{h_1,\ldots,h_n}x(g)_s-D_{h_1,\ldots,h_n}x(\tilde{g})_s\right)\mathrm{d}g_s,\end{aligned}$$

with $w(h_1,\ldots,h_n;g)$ defined above. Using (10) and the induction hypothesis, we deduce easily that the map $g\mapsto D^nx$ is continuous, and the map $g\mapsto x$ is n times Fréchet differentiable.

Step 2. Eq. (20) can be written in the following way:

$$\begin{aligned}\Phi_t^{i,i_1,\ldots,i_n}(r_1,\ldots,r_n) &= \gamma_t^{i,i_1,\ldots,i_n}(r_1,\ldots,r_n)\\ &\quad+\sum_{k_1=1}^{d}\int_{r_1\vee\ldots\vee r_n}^t\partial_{k_1}b^i(x_s)\,\Phi_s^{i,i_1,\ldots,i_n}(r_1,\ldots,r_n)\mathrm{d}s\\ &\quad+\sum_{k_1=1}^{d}\sum_{l=1}^{m}\int_{r_1\vee\ldots\vee r_n}^t\partial_{k_1}\sigma^{il}(x_s)\,\Phi_s^{i,i_1,\ldots,i_n}(r_1,\ldots,r_n)\mathrm{d}g_s^l,\end{aligned}\tag{32}$$

where

$$\gamma_t^{i,i_1,\ldots,i_n}(r_1,\ldots,r_n) = \sum_{i_0=1}^{n} A^i_{i_0,i_1,\ldots,i_0-1,i_0+1,\ldots,i_n}(r_{i_0},r_1,\ldots,r_{i_0-1},r_{i_0+1},\ldots,r_n)$$
$$+ \sum_{\substack{I_1\cup\cdots\cup I_\nu\\ \nu\geq 2}} \sum_{k_1,\ldots,k_\nu=1}^{d} \sum_{j=1}^{m} \int_{r_1\vee\cdots\vee r_n}^{t} \partial_{k_1}\ldots\partial_{k_\nu}\sigma^{ij}(x_s)$$
$$\times\, \Phi_s^{k_1,i(I_1)}(r(I_1))\cdots\Phi_s^{k_\nu,i(I_\nu)}(r(I_\nu))\mathrm{d}g_s^j,$$
$$+ \sum_{\substack{I_1\cup\cdots\cup I_\nu\\ \nu\geq 2}} \sum_{k_1,\ldots,k_\nu=1}^{d} \int_{r_1\vee\cdots\vee r_n}^{t} \partial_{k_1}\ldots\partial_{k_\nu}b^i(x_s)\ \Phi_s^{k_1,i(I_1)}(r(I_1))\ldots\Phi_s^{k_\nu,i(I_\nu)}(r(I_\nu))\mathrm{d}s.$$

Notice that the function $\Phi_t^{i,i_1,\ldots,i_n}(r_1,\ldots,r_n)$ is symmetric in $(i_1,r_1),\ldots,(i_n,r_n)$ for any t,i. As we did for Eq. (24) (see Proposition 9), we can show by induction that there exists a unique solution of (32) which is Hölder continuous of order $1-\alpha$ in all its variables.

Step 3. Let us show Eq. (19).We again proceed by induction. (19) is true for $n=1$ by (15). Assume that it is true up to the rank $n-1$. For any subset $K=\{i_1,\ldots,i_\nu\}$ of $\{1,\ldots,n\}$, we define $|K|$ as its cardinal and $\mathrm{d}h^K(r(K))=\mathrm{d}h^{i_1}(r_{i_1})\cdots\mathrm{d}h^{i_\nu}(r_{i_\nu})$. Using Fubini's theorem (by the previous step, the integrals are Riemann–Stieltjes ones) and the induction hypothesis we have

$$\int_0^t\left\{\int_0^t\cdots\int_0^t B^i_{i_1,\ldots,i_n}(r_1,\ldots,r_n;s)\mathrm{d}h_1^{i_1}(r_1)\cdots\mathrm{d}h_n^{i_n}(r_n)\right\}\mathrm{d}s$$
$$=\int_{r_1\vee\cdots\vee r_n}^{t}\left\{\sum_{I_1\cup\cdots\cup I_\nu}\sum_{k_1,\ldots,k_\nu=1}^{d}\partial_{k_1}\ldots\partial_{k_\nu}b^i(x_s)\right.$$
$$\times\left(\int_{[0,T]^{|I_1|}}\Phi_s^{k_1,i(I_1)}(r(I_1))\mathrm{d}h^{i(I_1)}(r(I_1))\right)$$
$$\left.\times\cdots\times\left(\int_{[0,T]^{|I_\nu|}}\Phi_s^{k_\nu,i(I_\nu)}(r(I_\nu))\mathrm{d}h^{i(I_\nu)}(r(I_\nu))\right)\right\}\mathrm{d}s$$
$$=\int_{r_1\vee\cdots\vee r_n}^{t}\left\{\sum_{I_1\cup\cdots\cup I_\nu}\sum_{k_1,\ldots,k_\nu=1}^{d}\partial_{k_1}\ldots\partial_{k_\nu}b^i(x_s)D^{i(I_1)}x_s^{k_1}\cdots D^{i(I_\nu)}x_s^{k_\nu}\right\}\mathrm{d}s.$$

Similar computations for the other terms of (20) yield that

$$t\mapsto\sum_{i_1,\ldots,i_n=1}^{m}\int_0^t\cdots\int_0^t\Phi_t^{i,i_1,\ldots,i_n}(r_1,\ldots,r_n)\mathrm{d}h_1^{i_1}(r_1)\mathrm{d}h_2^{i_2}(r_2)\cdots\mathrm{d}h_n^{i_n}(r_n)$$

is a solution of (31) and then it is equal to $D_{h_1,\ldots,h_n}x_t^i$ by the existence and uniqueness result for such equations. ■

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